# ERROR IN "THE EINSTEIN-CARTAN THEORY: THE MEANING AND CONSEQUENCES OF TORSION

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Abstract. Great stuff.

## 1. Assumptions and notation

Spacetime is endowed with an $SO(3,1)$ principal bundle of frames, with an $so(3,1)$-valued connection form $\omega^a_b$.$\nabla$ denotes the covariant derivative. $e_a$ is an orthonormal moving frame, $e^a$ its dual. $x^\mu$ is a local coordinate system. We have, using the conventions as in Ref. [1],

$$e_a = e^\mu_a\,\partial_\mu, \quad \partial_\mu = e^a_\mu\,e_a, \tag{1.1}$$

$$\nabla_\mu e_a = \omega^b{}_{a\mu} e_b, \quad \nabla_\mu \partial_\nu = \Gamma^\sigma_{\nu\mu}\,\partial_\sigma. \tag{1.2}$$

*Remark* 1.1. Notice that the form index is the **last index**. Thus, for example,

$$\omega^b{}_a = \omega^b{}_{a\mu}\,dx^\mu = \omega^b{}_{ac}\,e^c. \tag{1.3}$$

*We consider the special case od vanishing connection form* $\omega^a{}_b = 0$. Therefore

$$\nabla_\mu\,e_a = 0, \quad \nabla_\mu\,e^a = 0. \tag{1.4}$$

Then

$$0 = \nabla_\mu\,\partial_\nu = \nabla_\mu\,(e^a_\nu\,e_a) = (\partial_\mu\,e^a_\nu)\,e_a = (\partial_\mu\,e^a_\nu)\,e^\sigma_a\,\partial_\sigma. \tag{1.5}$$

Therefore

$$\Gamma^\sigma_{\nu\mu} = (\partial_\mu\,e^a_\nu)\,e^\sigma_a. \tag{1.6}$$

In particular the curvature vanishes.

### 1.1. Calculating torsion. $T^\sigma_{\mu\lambda}$ and $T^a_{\mu\nu}$:

$$T^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\nu} - \Gamma^\sigma_{\mu\nu} = (\partial_\mu\,e^a_\nu - \partial_\nu\,e^a_\mu)\,e^\sigma_a, \tag{1.7}$$

$$T^a_{\mu\nu} = \partial_\mu\,e^a_\nu - \partial_\nu\,e^a_\mu. \tag{1.8}$$

## 2. Mathisson-Papapetrou equation - Eq. (3.31) in the paper

### 2.1. Errors in deriving the consequences of the equation.
Assuming spin zero:

$$(D_\tau\,p_a)\,e^a_\mu = p_a\,\dot{z}^\lambda\,T^a_{\mu\lambda}, \tag{2.1}$$

where $D_\tau$ is the covariant derivative along the path. This is the famous Mathisson-Papapetrou equation for spin $S = 0$, in an explicit form. Written in terms of a more abstract fiber bundle language it can found e.g. in Ref. [2, Eq. (2.11)].[1]

*Explanation*: Suppose, in the coordinate system $x^\mu$, the trajectory is given by a function $\tau \mapsto x^\mu(\tau)$. Following the notation of the paper, I denote the derivative $dx^\mu/dt$ by $\dot{z}^\mu$. it is a section of the tangent bundle over the trajectory. There is also a variable $p$, section of the cotangent bundle over the trajectory, the "momentum": $p = p_a\,e^a$. In an orthonormal frame we lower and rise indices with the flat space metric tensor $\eta$, that gives the isomorphism between the tangent and cotangent space. $D_\tau\,p_a$ should be understood as

$$D_\tau\,p_a = (\dot{z}^\mu\,\nabla_\mu p)_a. \tag{2.2}$$

[1] Warning: Sternberg [2] denotes by $\omega$ not the connection form, but the soldering form, in our notation the tetrad $e^a_\mu$. The connection form is denoted there as $\theta$.

Since we have assumed that $\omega^a{}_b = 0$. the covariant derivative of $p_a$ reduces to the ordinary derivative, therefore

$$(d\,p_a/d\tau)\,e^a_\mu = p_a\,\dot{z}^\lambda\,T^a_{\mu\lambda}. \tag{2.3}$$

**Additional assumption - top of Sec. 3.4, p. 21 of the paper** We assume

$$p^a = e^a_\kappa\,\dot{z}^\kappa, \tag{2.4}$$

therefore

$$p_a = e_{a\kappa}\,\dot{z}^\kappa, \tag{2.5}$$

where

$$e_{a\kappa} = \eta_{ab}\,e^b_\kappa,\ \eta = \mathrm{diag}(-1,1,1,1). \tag{2.6}$$

In other words, momentum is proportional to velocity. Proportionality factor is constant. Then $\tau$ can be considered as being the arc length. Notice that, with the assumption (2.4),

$$p_a e^a_\mu = \dot{z}_\mu. \tag{2.7}$$

**Calculating the RHS of Eq. (2.3)**

$$(d\,p_a/d\tau)\,e^a_\mu = p_a\,\dot{z}^\lambda\,(\partial_\mu\,e^a_\lambda - \partial_\lambda\,e^a_\mu), \tag{2.8}$$

$$= p_a\,\dot{z}^\lambda\,\partial_\mu\,e^a_\lambda - p_a\,\dot{z}^\lambda\,\partial_\lambda\,e^a_\mu. \tag{2.9}$$

We will now rewrite the second term. First of all we write

$$\dot{z}^\lambda\,\partial_\lambda\,e^a_\mu = \frac{d}{d\tau}e^a_\mu(z(t)). \tag{2.10}$$

The we use the product rule for derivatives as $fg' = (fg)' - f'g$ to rewrite the second term in (2.9) as

$$p_a\,\dot{z}^\lambda\,\partial_\mu\,e^a_\lambda = p_a\frac{d}{d\tau}e^a_\mu(z(t)) = \frac{d}{d\tau}(p_a e^a_\mu) - \frac{dp_a}{d\tau}e^a_\mu. \tag{2.11}$$

From Eq. (2.5) we have

$$p_a e^a_\mu = e_{a\kappa}\,\dot{z}^\kappa e^a_\mu = g_{\kappa\mu}\dot{z}^\kappa = \dot{z}_\mu. \tag{2.12}$$

Thus, Eq. (2.11) becomes

$$p_a\,\dot{z}^\lambda\,\partial_\mu\,e^a_\lambda = \ddot{z}_\mu - \frac{dp_a}{d\tau}e^a_\mu. \tag{2.13}$$

Taking into account the minus sign in front of the second term of the RHS of Eq. (2.9) the RHS becomes:

$$(d\,p_a/d\tau)\,e^a_\mu = p_a\,\dot{z}^\lambda\,\partial_\mu\,e^a_\lambda - \ddot{z}_\mu + \frac{dp_a}{d\tau}e^a_\mu. \tag{2.14}$$

The last term on the RHS is the same as LHS - they cancel. Therefore we obtain

$$\ddot{z}_\mu = p_a\,\dot{z}^\lambda\,\partial_\mu\,e^a_\lambda = p^a\,\dot{z}^\lambda\,\partial_\mu\,e_{a\lambda} = e^a_\kappa\,\dot{z}^\kappa\dot{z}^\lambda\,\partial_\mu\,e_{a\lambda}, \tag{2.15}$$

that is only the first term of Eq. (3.50) in the paper [1].
Additionally, notation $D_\mu\,e_{a\lambda}$ etc. in Eq. (3.50) of the paper is misleading, since with our assumptions the covariant derivatives of the frame and coframe vanish. The error in the paper [1] is in the transition (3.46)-(3.47) on p. 21. The author uses there the formula

$$e_{a\kappa}\ddot{z}^\kappa\,e^a_\mu = \ddot{z}_\mu. \tag{2.16}$$

Then $\ddot{z}_\mu$ is interpreted as the derivative of $\dot{z}_\mu$. But this is incorrect. Derivative of $\dot{z}_\mu$ should be calculated as follows:

$$\ddot{z}_\mu = d\dot{z}_\mu/d\tau = d(e_{a\kappa}\dot{z}^\kappa\,e^a_\mu) = e_{a\kappa}(d\dot{z}^\kappa/d\tau)\,e^a_\mu + \dot{z}^\kappa d(e_{a\kappa}e^a_\mu). \tag{2.17}$$

The second term is neglected in the paper. Additionally, the equations should not be solved for the variable $z_\mu$, as there is no such variable. Instead, one should write it in terms, and solve it, for the variable $z^\mu$.

2.2. **General conceptual error.** Even if the errors discussed in the previous section are corrected, the paper would still give an impression that torsion plays some particular role for the motion of zero spin particles. But, in fact, it does not. The following reasoning demonstrates that the *trajectories of spin zero particles are simply geodesics of the Levi-Civita connection of the metric* $g_{\mu\nu}$. *Assumption of zero curvature is not needed for that.*
Since we are going to lower and rise indices in the torsion two-form, it is important to write the indices in proper position. Thus for the torsion I will write $T^a{}_{bc}$. Let us rewrite Eq. (2.1) to tally in terms of indices $b, c, ...$ referring to the orthonormal frame. Notice that then

$$\dot{z}^a = p^a. \tag{2.18}$$

Thus we have

$$\frac{Dp_a}{d\tau} = p_c p^b T^c{}_{ab}, \tag{2.19}$$

which is the same as

$$\frac{Dp_a}{d\tau} = p^c p^b T_{cab}. \tag{2.20}$$

Recall the definition of the contorsion tensor $K^a{}_{bc}$:[2]

$$\Gamma^a_{bc} = \hat{\Gamma}^a_{bc} + K^a{}_{bc}, \tag{2.21}$$

where $\hat{\Gamma}^a_{bc}$ are the coefficients of the Levi-Civita connection. Recall also the connection between the contorsion and torsion (cf. e.g. Ref. [3, p. 10, Eq. (1.63)], or Wikipedia, `http://en.wikipedia.org/wiki/Contorsion_tensor`[3]

$$K^a{}_{bc} = \frac{1}{2}(T_b{}^a{}_c + T_c{}^a{}_b - T^a{}_{bc}). \tag{2.22}$$

Taking into account the antisymmetry of the torsion tensor we find that

$$p^c p^b T_{cab} = p^c p^b K_{abc}. \tag{2.23}$$

On the LHS of Eq. (2.20) we have the covariant derivative along the trajectory, which expands as

$$\frac{Dp_a}{d\tau} = \frac{dp_a}{dt} - p^b \Gamma^c{}_{ab} p_c = \frac{dp_a}{dt} - p^b \hat{\Gamma}^c_{ab} p_c - p^b K^c{}_{ab} p_c. \tag{2.24}$$

The first two terms is just the covariant derivative $\frac{\hat{D}p_a}{d\tau}$ along the path with respect to the Levi-Civita connection. The second term we rewrite as

$$-p^b K^c{}_{ab} p_c = -p^b p^c K_{cab} = p^b p^c K_{acb}, \tag{2.25}$$

where we have used the fact that $K^c{}_{ab}$ is a one-form (form index $b$) with values in the the Lie algebra $so(3,1)$, therefore $K_{cab} = -K_{acb}$. Thus

$$\frac{Dp_a}{d\tau} = \frac{\hat{D}p_a}{d\tau} + p^b p^c K_{acb}. \tag{2.26}$$

Returning to Eq. (2.20) we see that the terms with contorsion on both sides of the equation cancel, and what remains is simply

$$\frac{\hat{D}p_a}{d\tau} = 0, \tag{2.27}$$

which (recall that $p^a = \dot{z}^a$ ) is the equation for arc-parametrized geodesics of the Levi-Civita connection.

## References

[1] Grabowski-Laskos, P., "The Einstein-Cartan theory: the meaning and consequences of torsion", thesis 2009 `https://th.if.uj.edu.pl/~plg/ph.pdf`
[2] Sternberg, S., Magnetic Moments and General Covariance, Lett. Math. Phys. **9** (1985), 35-42
[3] Aldrovandi, R., Pereira, J. G., *Teleparallel Gravity*, Springer 2013

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[2] $\Gamma^a_{bc}$ is the same as $\omega^a_{bc}$ - the Lie algebra valued connection form.
[3] Wikipedia uses different indexing convention though.