Notes on Clifford Algebras

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Abstract

My notes while studying general Clifford algebras

1 Introduction

1.1 Preliminaries

A reasonably general formulation of the theory of Clifford algebras starts with the definition of the Clifford algebra of a module over a ring, equipped with a quadratic form. A not necessarily symmetric and, in general, degenerate, bilinear form can also appear within this theory. Later on modules are replaced by vector spaces over a field of characteristic zero. We start with the definitions, where we follow the references [5, 12]

Definition 1.1 (Ring)**.** *A ring is a set R with two laws of composition, one denoted additively and the other multiplicatively, which satisfy the following conditions:*

1. *The elements of R form a commutative group under addition;*
2. *The elements of R form a monoid under multiplication;*
3. *If a, b, c are elements of R, we have*

$$a(b + c) = ab + ac,\ (a + b)c = ac + bc.$$

That R is a monoid under multiplication means that

1. *$(ab)c = a(bc)$ for all $a, b, c \in R$ (associativity),*

2. *There is an element* $1 \in R$ *such that* $1a = a1 = a$ *for all* a *in* R *(that is* 1 *is the multiplicative identity (neutral element).*

A ring containing at least two elements, in which every nonzero element a *has a multiplicative inverse* a^{-1} *is called a* division ring *(sometimes also called a "skew field"). A commutative division ring is called a* field.

Definition 1.2 (Characteristic)**.** *Let* R *be a ring with unit element* 1*. The* characteristic *of* R *is the smallest positive number* n *such that*

$$\underbrace{1 + ... + 1}_{n \text{ summands}} = 0.$$

If such a number does not exist, the characteristic is defined to be 0.

We notice that the above condition is equivalent to

$$\underbrace{\alpha + ... + \alpha}_{n \text{ summands}} = 0$$

for every $0 \neq \alpha \in R$.

In applications to Clifford algebras R **will be always assumed to be commutative**. Ultimately R will become the field of real or of complex numbers, but for a while it costs us nothing to be more general. The notation and definitions below follow closely those in Ref. [6].

Definition 1.3 (Module)**.** *Let* R *be a commutative ring. A* module *over* R *is a set* M *such that*

1. M *has a structure of an additive group,*
2. *For every* $\alpha \in R$, $a \in M$ *an element* $\alpha a \in M$ *called scalar multiple is defined, and we have*
   - *i)* $\alpha(x + y) = \alpha x + \alpha y$,
   - *ii)* $(\alpha + \beta)x = \alpha x + \beta x$,
   - *iii)* $\alpha(\beta x) = (\alpha\beta)x$,
   - *iv)* $1 \cdot x = x$.

Remark 1.4. *Normally one would distinguish between left modules and right modules, where multiplication by scalars (element of the ring* R*) is defined from the left or from the right. But since we will assume that* R *is commutative, there is no necessity to distinguish between left and right modules. Indeed, in a left module, multiplying* x *first by* α*, then by beta, we would get* $\beta\alpha x$*, thus getting* x *multiplied by* $\beta\alpha$*. In a right module doing the same we would get* $x\alpha\beta$*, thus* x *multiplied by* $\alpha\beta$*. In a commutative module* $\alpha\beta = \beta\alpha$*, therefore it does not matter whether we write the multiplication on the left or on the right.*

1.1.1 Vector spaces

Definition 1.5 (Vector space). *If R is a division ring, then a module M over R is called a* vector space.

In general one would consider left and right vector spaces, but since we assume that R is commutative, it is not necessary to distinguish between the two cases. In linear algebra one shows that every vector space has a basis, possibly infinite, of linearly independent vectors. Moreover, every two bases have the same cardinal number called the dimension of the vector space - cf. e.g. [5, p. 103]. Every system of linearly independent vectors can be extended to a basis. In particular for every nonzero vector there exists a linear functional on the space (element of the dual space) that takes a nonzero value on this vector. Clifford algebras are usually studied with a restriction to finite dimensional vector spaces. But such a restriction is not necessary at the very beginning, for the study of many important general properties of Clifford algebra.

1.1.2 Algebras

Definition 1.6 (Associative algebra with identity called simply hereafter algebra). *An algebra A over R is a module over R with a multiplication which makes A a ring and satisfying*

$$\alpha(xy) = (\alpha x)y = x(\alpha y),\ (x, y \in A,\ \alpha \in R).$$

Notice that it follows from the definition above, the part where it is said that A is a ring, that the algebra will be always assumed to contain a neutral element, usually denoted as 1.

A subset B of an algebra A is called a *subalgebra* if for any x, y from B, α in R, also $\alpha x, x + y, xy$ are in B, and if B contains the unit 1 of A. A subset S of an algebra A is called a set of generators if A is the smallest subalgebra of A containing S and the unit 1 of A.

1.1.3 Tensor algebra

Definition 1.7 (Tensor algebra). *Let M be a module over R. An algebra T is called a tensor algebra over M (or "of M") if it satisfies the following* universal property

1. *T is an algebra containing M as a submodule, and it is generated by M,*

2. *Every linear mapping λ of M into an algebra A over R, can be extended to a homomorphism θ of T into A.*

Note 1.8 (Chevalley's construction of the tensor algebra)**.** *In all standard textbooks, see e.g. [3, 6, 12], the above characterisation of the tensor algebra of a module is always completed by a prove of its existence, i.e. by its construction. Chevalley [6] does it in an original way, using the construction of a free algebra as follows.*

Step 1 *First of all given any set $\{x_i\}_{i\in I}$ indexed by an index i in some indexing set I, we can construct an algebra in which this set is the set of linearly and algebraically independent generators. The construction goes as follows. We consider the set Σ of all finite sequences of elements of I. In Σ we include also the empty sequence σ_0 containing no elements from I. With σ_0 we associate the symbol "1". It will become the unit element of our algebra. From theorems of linear algebra we know that there exists a module which has a basis that is equipotent to the set Σ. In other words, there exists a module F in which there is a basis that can be indexed by means of the elements of the set Σ. Given an element $\sigma \in \Sigma$, that is a finite sequence of elements of I we have $\sigma = \{i_1, ..., i_n\}$. Let $y_{\sigma \sigma\in\Sigma}$ be the basis in F. To define the algebra multiplication in F we only need to specify the multiplication of the basis elements. This is defined in a natural way as a juxtaposition $y_\sigma y_{\sigma'} = y_{\sigma\sigma'}$. At the end we can replace every symbol i with the corresponding element of the set $\{x_i\}_{i\in I}$. In this way we obtain* the free algebra with the set $\{x_i\}_{i\in I}$ as the set of generators. *Notice that it follows automatically that the symbol "1" becomes the unit of our algebra, as a juxtaposition of the empty set σ_0 with any σ is σ.*

Step 2 *Let now M be a module. We will construct the tensor algebra $T(M)$ of M. First we consider M as a set, ignoring its module structure. Then we build the free algebra F with M as the set of generators. And now we take into account the existing module structure of M by dividing F by an appropriate two sided ideal as follows. In F we have the algebra structure introduced by its construction. In order to distinguish between the linear operations within F from those within M we denote the addition and subtraction in F by the symbols $\dot{+}$ and $\dot{-}$, and multiplication by scalars by $\alpha \cdot x$. Thus, right now, in F we have, for instance, if x, y are in M, then $x + y \in M$ but, in general, $x \dot{+} y \notin M$, and also $\alpha x \in M$ but $\alpha \cdot x \notin M$. To build the tensor algebra over M we need, for $x, y \in M$, to have $x \dot{+} y = x + y$ and $\alpha \cdot x = \alpha x$. To this end let S be the set of all elements of the forms:*

$$x \dot{+} y \dot{-} (x + y),\ (x, y \in M),$$

and

$$\alpha \cdot x \dot{-} (\alpha x)\,(\alpha \in R, x \in M),$$

and let $\mathcal{T}$ be the two sided ideal in F generated by S. The tensor algebra $T(M)$ of M is then defined as the quotient $F/\mathcal{T}$. Chevalley then shows that $T(M)$ so constructed has the universal property described in Definition 1.7

Let M be a module over R and let $T(M)$ be its tensor algebra. The multiplication within the algebra $T(M)$ inherited from the algebra F is denoted $\otimes$. Since the ideal $\mathcal{T}$ is generated by elements homogeneous of grade 1 in M, the resulting algebra $T(M)$ is also graded. We have

$$T(M) = \bigoplus_{p=0}^{\infty} T^p M, \tag{1}$$

where

$$T^p M = M^{\otimes p} = \underbrace{M \otimes ... \otimes M}_{p \text{ factors}}. \tag{2}$$

It is understood here that $T^0 M = R$ and $T^1 M = M$. The tensor algebra is a graded and associative (but non-commutative) algebra, with unit $1 \in R$. The fact that $T(M)$ is a graded algebra means that for any $x \in T^p M, y \in T^q M$ the product xy is in $T^{p+q}M$ for all $p, q = 0, 1, \cdots$. Sometimes it is convenient to consider $T^p M$ for $p < 0$ as consisting of the zero vector only.

1.1.4 Quadratic forms

Given a module M over a ring R we will define now quadratic form on M. There are two definitions possible, one more general than the other one if general rings with any characteristic are being considered. Bourbaki [3] and Chevalley [6] use the more general definition adapted to a general case. Below I will give an example of how careful one has to be in a general case, I will closely follow the monograph by Helmstetter [9].

Definition 1.9 (Quadratic form I)**.** *Let M be a module over a commutative ring R. A mapping $q : M \to R$ is called a quadratic form on M if the following conditions are satisfied:*

1.
$$q(\alpha x) = \alpha^2\, q(x)\, \text{for all } \alpha \in R,\, x \in M, \tag{3}$$

2. *There exists a bilinear form $\Phi(x, y)$ on M such that for all $x, y \in M$ we have*
$$\Phi(x, y) = q(x + y) - q(x) - q(y). \tag{4}$$

We say that the bilinear form Φ is associated with the quadratic form q. Sometimes Φ is also called the polar form of q. *It follows from its very definition that Φ is* symmetric*: $\Phi(x, y) = \Phi(y, x)$ for all $x, y \in M$.*

We can combine Eqs. (4) and (3) into:

$$q(\alpha x + \beta y) = \alpha^2 q(x) + \beta^2 q(y) + \alpha\beta\Phi(x, y). \tag{5}$$

The short discussion of consequences given below is taken directly from Ref. [9].

Note 1.10. *From the very definition we find that*

$$\Phi(x, x) = q(2x) - 2q(x) = 4q(x) - 2q(x) = 2q(x). \tag{6}$$

It follows that if R is of characteristic 2, then $\Phi(x, x) = 0$ for all $x \in R$. Such a form is called alternate*. In that case, since also $\Phi(x + y, x + y) = 0$, we have that*

$$\begin{aligned} 0 = \Phi(x + y, x + y) = \Phi(x, x) + \Phi(x, y) + \Phi(y, x) + \Phi(y, y) \\ = \Phi(x, y) + \Phi(y, x), \end{aligned} \tag{7}$$

so that in this case the form Φ is antisymmetric $\Phi(x, y) = -\Phi(y, x)$.

Getting back to a general characteristic, we may also notice at this point that *if the mapping $x \mapsto 2x$ is surjective* in M, then the form Φ determines q. Indeed, setting $y = 2x$ we get $q(y) = q(2x) = 4q(x) = 2\Phi(x, x)$. We also observe that the quadratic form q is determined by the associated bilinear form Φ when the mapping $\alpha \mapsto 2\alpha$ is injective in R, in other words if multiplication by $\frac{1}{2}$ makes sense in R. In that case we can solve Eq. (6) to obtain $q(x) = \frac{1}{2}\Phi(x, x)$.

In applications to Clifford algebras, unless we are interested in very special cases like characteristic 2, it is more convenient to use a little bit different definition of a quadratic form, as given, for instance, in Ref. [13, p. 199]:

Definition 1.11 (Quadratic form II)**.** *Let M be a module over a commutative ring R. A function $q : M \to R$ is called a quadratic form if there exists a bilinear form $F : M \times M \to R$ such that*

$$q(x) = F(x, x). \tag{8}$$

It follows from this last definition that the condition in Eq.(3) is then automatically satisfied, and also the condition in Eq.(4) is automatically satisfied with

$$\Phi(x, y) = F(x, y) + F(y, x). \tag{9}$$

Remark 1.12. *If the module M admits a basis (in particular, when it is a vector space), then given a quadratic form q as in Def. 1.9 one can always construct a bilinear form F (in general a non symmetric one) such that $q(x) = F(x,x)$ (cf. eg. Ref. [4, Proposition 2, p. 55]). It is instructive to understand the idea of the proof (taken from Ref. [4, Proposition 2, p. 55]).* [1] *Of course if the field R admits division by 2, we can use Eq. (6) and simply set $F(x,y) = \frac{1}{2}\Phi(x,y)$. In particular the rest of this remark is irrelevant for vectors spaces over reals or complex number fields*

Let q be a quadratic form on a vector space M, and let Φ be the associated bilinear form. We start with noticing that M, being a vector space, has a basis $\{e_i\}_{i\in I}$. By the well-ordering theorem every set can be well ordered, and we will assume that the index set I is well ordered. Since $\{e_i\}_{i\in I}$ is a basis, every bilinear form F is uniquely determined by the coefficients f_{ij}, $i,j \in I$. Let Φ be the bilinear form associated to q. We first observe that if $\{\alpha_i\}_{i\in I}$ is any family of elements of R with only a finite number of $\alpha_i \neq 0$, then

$$q(\sum_i \alpha_i e_i) = \sum_i \alpha_i^2 q(e_i) + \sum_{\{i,j\}} \alpha_i\alpha_j \Phi(e_i, e_j), \tag{10}$$

where the last sum is over all two-element subsets of I. [2]

It is understood that each sum is over a finite set determined by non-zero α_i-s. We prove Eq. (10) by induction with respect to the number n of nonzero coefficients α_i. If there are only two nonzero coefficients, then (10) follows from Eq. (5), i.e. from the definition of the quadratic form 1.9. Assume now that Eq. (10) holds for subsets $\{i_1, ..., i_n\}$ of n non-zero coefficients α_i, and let us add another non-zero coefficient $\alpha_{i_{n+1}}$. Then

$$\begin{aligned} &q(\alpha_{i_1}e_{i_1} + ... + \alpha_{i_{n+1}}e_{i_{n+1}}) = q\left((\alpha_{i_1}e_{i_1} + ... + \alpha_{i_n}e_{i_n}) + \alpha_{i_{n+1}}e_{i_{n+1}}\right) \\ &= q(\alpha_{i_1}e_{i_1} + ... + \alpha_{i_n}e_{i_n}) + q(\alpha_{i_{n+1}}e_{i_{n+1}}) + \Phi(\alpha_{i_1}e_{i_1} + ... + \alpha_{i_n}e_{i_n}, \alpha_{i_{n+1}}e_{i_{n+1}}). \end{aligned}$$

Using now the quadratic form property, in particular for the sum of two elements, the assumed property for the sum of n elements, as well as linearity of Φ in the first argument leads to the desired result. Nowhere we needed to assume that the basis $\{e_i\}_{i\in I}$ is finite.

Given a quadratic form q we can now define a bilinear form F satisfying

[1] The proof can be also found in Ref. [I.2.2, p. 76][6], but with unnecessary assumption that M is finite dimensional.

[2] Thus if a_{i_1} and a_{i_2} are nonzero, with $i_1 < i_2$, then only $\Phi(e_{i_1}, e_{i_2})$ enters the sum, and not $\Phi(e_{i_2}, e_{i_1})$ because $\{i_2, i_1\}$ is the same subset as $\{i_1, i_2\}$

$q(x) = F(x, x)$ by defining its coefficients f_{ij}, $i, j \in I$, as follows:

$$\begin{aligned} f_{ii} &= q(e_i), & (11) \\ f_{ij} &= \Phi(e_i, e_j),\ i < j, & (12) \\ f_{ij} &= 0,\ i > j. & (13) \end{aligned}$$

We now check that $q(x) = F(x, x)$ for every x in M. If $x \in M$ then $x = \sum_i \alpha_i e_i$, with only a finite number of non-zero terms in the sum. Therefore, using Eq. (10) we have

$$q(x) = \sum_i \alpha_i^2 q(e_i) + \sum_{i<j} \alpha_i \alpha_j \Phi(e_i, e_j). \tag{14}$$

On the other hand, from bilinearity of F we have

$$F(x, x) = F(\sum_i \alpha_i e_i, \sum_j \alpha_j e_j) = \sum_i \alpha_i^2 f_{ii} + \sum_{i \neq j} \alpha_i \alpha_j f_{ij} = q(x)$$

from the definition of the coefficients f_{ij} above (because $f_{ij} = 0$ for $i > j$) .

1.1.5 Diagonalization of symmetric bilinear forms

When studying Clifford algebras it is often convenient to use particularly nice properties of orthogonal bases for symmetric bilinear forms. Such a basis always exists for finite dimensional vector spaces over the field of characteristic different from 2, and it is instructive to look at the proof of the proposition below (taken from Ref. ([8, p. 362]), cf. also ([10]).

Proposition 1.13. *Let M be a finite dimensional vector space over a field of characteristic $\neq 2$ and let F be a symmetric bilinear form on M. Then there exists a basis $\{e_i\}$, $(i = 1, ..., n)$ in M consisting of mutually orthogonal vectors: $F(e_i, e_j) = 0$ for $i \neq j$. In other words F is diagonalizable.*

Proof. The proof is by induction with respect to the dimension n of the vector space. The statement is trivially true for $n = 1$, since in this case the set $i \neq j$ is empty. Suppose the statement holds for vector spaces of dimension $n - 1$ or less. We will show that then it holds also for dimension n. For this we will need a little auxiliary results, and it is in the proof of this auxiliary result we will use the fact that the characteristic is $\neq 2$. Namely we first need to show that if the symmetric bilinear form is nontrivial, i.e. $F \neq 0$, then there always exists a vector x such that $F(x, x) \neq 0$. The fact that $F \neq 0$ is equivalent to saying that there exist vectors u, v for which $F(u, v) \neq 0$. If $F(u, u) \neq 0$ or $F(v, v) \neq 0$, we are done, but if

$F(u,u) = 0$ and $F(v,v) = 0$, then $x = u+v$ does the job. Indeed then $F(x,x) = F(u,u) + F(v,v) + F(u,v) + F(v,u) = 2F(u,v)$ since we have assumed that F is symmetric. But, since we also assume that the filed is not of characteristic 2, then $2 \neq 0$, and therefore $F(x,x) \neq 0$.

Let us return now to the proof of the main statement, assuming M n-dimensional. If $F = 0$ any basis does the job. Let us therefore assume that $F \neq 0$. Then, as we have just shown, there exists a vector x such that $F(x,x) \neq 0$. Evidently $x \neq 0$. Then we define W as the following subspace of M (the orthogonal complement of x).

$$W = \{w \in M : F(x,w) = 0\}. \tag{15}$$

Evidently W is a vector space that does not contain x. Moreover we have that every vector $v \in M$ can be uniquely written in the form $v = w + \alpha x$, where $w \in W$ and α is a scalar. For if v is in W we set $\alpha = 0$ and if $v \notin W$, then $F(x,v) \neq 0$, and it is enough to set $\alpha = F(x,v)/F(x,x)$ and define $w = v - \alpha x$. Then automatically $F(w,x) = 0$ i.e. $w \in W$, and $v = w + \alpha x$.

If $\{e_i\}$ is a basis in W, then $\{e_i\} \cup \{x\}$ is a basis in M. Therefore W is $n-1$ dimensional and, by the induction hypothesis, there exists a basis $e_1, ..., e_{n-1}$ in W diagonalizing F. But then e_i together with x is a basis in M, and it is diagonalizing F, since $e_i \in W$ and therefore, by the definition of W, $F(x, e_i) = 0$ for $i = 1, ..., n$. $\square$

1.1.5.1 Degenerate and nondegenerate bilinear forms

With the assumptions and notation as above, let F be a bilinear form on M, *but not necessarily symmetric*. When F is not necessarily symmetric, there are two possible definitions of a *degenerate bilinear form*:

(i) There exists $y \in M$, $y \neq 0$ such that $F(x,y) = 0\, \forall x \in M$;

(ii) There exists $x \in M$, $x \neq 0$ such that $F(x,y) = 0\, \forall y \in M$;

But in fact the two conditions are equivalent, and each of them is equivalent to the conditions that the matrix $\mathbf{F}_{ij} = F(e_i, e_j)$ is not invertible.

Indeed (i) is equivalent to: there exists $y \in M$, $y \neq 0$, such that $F(e_i, y) = 0$ for all $i = 1, ..., n$. Let us write $y = \sum_{j=1}^n y^j e_j$. Then $F(e_i, y) = 0$ can be written as $\sum_{j=1}^n F(e_i, e_j) y^j = 0$, or, in matrix notation, $\mathbf{F}\mathbf{y} = 0$, which is another way of saying that the matrix $\mathbf{F}$ is not invertible. The condition (ii) would lead to the same conclusion, but for the transposed matrix. But the matrix is invertible if and only if the transposed is invertible (inverse of the transpose is the transpose of the inverse [3, p. 350]), which shows the equivalence of (i) and (ii). The bilinear form that is not degenerate is called *nondegenerate*.

Remark 1.14. *We notice that a symmetric bilinear form F and an* orthogonal basis e_i*,* F *is non-degenerate if and only if all the diagonal elements* $F(e_i, e_i)$ *are non-zero. In fact, if* F *is nondegenerate, then all* $F(e_i, e_i)$ *must be non zero, since if one of them vanishes,* $F(e_i, e_i) = 0$*, then this* e_i *is orthogonal to all vectors in* M*. Conversely, if* F *is degenerate and there exists a non-zero* x *such that* $F(x, e_i) = 0$ *for all* i*, then one of the terms* $F(e_i, e_i)$ *must be zero. Indeed writing* $x = \sum_j x_j e_j$ *we find that* $0 = F(x, e_i) = \sum_j x_j F(e_j, e_i) = x_i F(e_i, e_i)$*, because of the orthogonality of the basis. If one of the coefficients* x_i *is non-zero, then* $F(e_i, e_i) = 0$*.*

1.2 Clifford algebras - definition

Let q be a quadratic form on M (see Def. 1.9), and let $J(q)$ be the two-sided ideal in $T(M)$ generated by elements of the form $x \otimes x - q(x)1$, where $x \in M \subset T(M)$. The ideal $J(q)$ consists of all finite sums of elements of the form $x_1 \otimes \ldots \otimes x_p \otimes (x \otimes x - q(x)1) \otimes y_1 \otimes \ldots \otimes y_q$, where $x, x_1, \ldots, x_p, y_1, \ldots, y_q$ are in M.

Definition 1.15 (Clifford algebra, cf. [6, p. 35])**.** *With* M *and* q *as above the quotient algebra* $\mathrm{Cl}(q) = T(M)/J(q)$ *is called the Clifford algebra associated to* M *and* q*.*

Denoting by $\pi_q : T(M) \to \mathrm{Cl}(q)$ the canonical mapping, $\pi_q(M)$ is a submodule of $\mathrm{Cl}(q)$ that generates $\mathrm{Cl}(q)$ as an algebra. Moreover, for all $x \in M$ we have

$$(\pi_q(x))^2 = q(x)1. \tag{16}$$

From $\pi_q(x+y)^2 - \pi_q(x)^2 - \pi_q(y)^2 = q(x+y) - q(x) - q(y) = \Phi(x, y)$ we find that

$$\pi_q(x)\pi_q(y) + \pi_q(y)\pi_q(x) = \Phi(x, y). \tag{17}$$

If M is a vector space, then the mapping $x \mapsto \pi_q(x)$ is injective (which will be shown later) and M can be identified with a linear subspace of $\mathrm{Cl}(q)$. In general it needs not be so. The case of $q = 0$ is special. The Ideal $J(q)$ is then generated by homogeneous elements $x \otimes x$ and the algebra $\mathrm{Cl}(0)$ is nothing but *the exterior algebra* $\Lambda(M)$ of M. All homogeneous elements of $J(0)$ are then of at least the degree 2, therefore no non-zero element of M can belong to this ideal. It follows that in this case the mapping $x \mapsto \pi(x)$ is an embedding and M can be always identified with the grade 1 subspace of $\mathrm{Cl}(0)$.

1.2.1 Universal property

The Clifford algebra $\mathrm{Cl}(q)$ defined above is characterized by a universal property analogous to the universal property characterizing the tensor algebra as defined in Definition 1.7.

Theorem 1.16 (Cf. e.g. [6, Theorem 3.1, p. 36])**.** *Assume that λ is a linear mapping from M into an algebra A with the property that $(\lambda(x))^2 = q(x)1$ for all x in M. Then there is a unique homomorphism ϕ of algebras over R, with units, such that for all x in M we have*

$$\lambda = \phi \circ \pi. \tag{18}$$

■

1.2.2 Main involution α and main anti-involution τ

It is by using this universal property that one defines the main involution α and the main anti-involution τ of $\mathrm{Cl}(q)$. To define α let λ be the map $\lambda : M \to \mathrm{Cl}(q)$ defined by $\lambda(x) = \pi(-x)$. Evidently

$$(\lambda(x))^2 = (\pi(-x))^2 = (-\pi(x))^2 = q(x).$$

Therefore λ defines ("extends to") a unique algebra homomorphism $\alpha : \mathrm{Cl}(q) \to \mathrm{Cl}(q)$ such that $\alpha(\pi(x)) = \pi(-x) = -\pi(x)$. It follows that

$$\alpha(\alpha(\pi(x))) = \pi(x)$$

thus $\alpha^2 \circ \pi = Id$ on M. From the uniqueness of the extension it follows then that $\alpha^2 = Id$, so that α is an involutive automorphism of $\mathrm{Cl}(q)$. It is called *the main involution*. Using the universal property in a similar but a somewhat different way one introduces *the main anti-involution* τ. Let $\mathrm{Cl}(q)^{op}$ denote the algebra opposite to $\mathrm{Cl}(q)$. That is $\mathrm{Cl}(q)^{op}$ is the same as $\mathrm{Cl}(q)$ as a linear space, but the multiplication is defined in the opposite order. The product xy in $\mathrm{Cl}(q)^{op}$ is the same as yx in $\mathrm{Cl}(q)$. But squares x^2 are evidently the same in both algebras. The identity map $\iota : x \mapsto x$ from $\mathrm{Cl}(q)$ to $\mathrm{Cl}(q)^{op}$ is an anti-homomorphism, $\iota(xy) = yx$. Consider the map $\lambda : M \to \mathrm{Cl}(q)^{op}$ defined as $\lambda(x) = \iota(\pi(x))$. Since the squares are the same in both algebras, we have that $(\lambda(x))^2 = q(x)1$. Therefore λ extends to an algebra homomorphism from $\mathrm{Cl}(q)$ to $\mathrm{Cl}(q)^{op}$. Composing this map with the inverse of ι we get $\tau : \mathrm{Cl}(q) \to \mathrm{Cl}(q)$. Arguing as in the previous case we deduce that $\tau^2 = Id$, therefore τ is an anti-automorphism of $\mathrm{Cl}(q)$. From the very definition we have that $\tau(\pi(x)) = \pi(x)$ for all $x \in M$. Since $\pi(M)$ generates $\mathrm{Cl}(q)$, this last property determines the anti-automorphism τ of $\mathrm{Cl}(q)$ uniquely.

Remark 1.17. *For a in* $\mathrm{Cl}(q)$ *we often write* a^τ *instead of* $\tau(a)$.

While the tensor algebra $T(M)$ is Z-graded, where Z stands for the Abelian group (under addition) of integers, the quotient algebra $\mathrm{Cl}(q) = T(M)/J(q)$ is only Z_2-graded. That is because the expressions $x \otimes x - q(x)1$ generating the ideal $J(q)$ are not grade homogeneous (unless $q = 0$, in which case $\mathrm{Cl}(q)$ is the exterior algebra of M).

1.2.2.1 The even subalgebra

There is another way of getting to the main automorphism α. Every element of the tensor algebra is a sum of even and odd tensors (that is tensors of even and odd degrees)

$$T(M) = T(M)_{\text{even}} \oplus T(M)_{\text{odd}}. \tag{19}$$

In the tensor algebra $T(M)$ the mapping $x \mapsto -x$ generates algebra automorphism, let us call it $\tilde{\alpha}$, that changes the sign of elements of $T(M)_{\text{odd}}$. Since the expressions $x \otimes x - q(x)$ generating the ideal $J(q)$ are invariant under the transformations $x \mapsto -x$, the automorphism $\tilde{\alpha}$ of the tensor algebra descends to the quotient algebra $\mathrm{Cl}(q)$. It maps $\pi_q(x)$ into $-\pi_q(x)$, therefore it coincides with the main automorphism α. It follows that α simply changes the sign of products of odd numbers of $\pi_q(x)\, x \in M$.

We now define $\mathrm{Cl}(q)_+ = \pi_q(T(M)_{\text{even}}), \mathrm{Cl}(q)_- = \pi_q(T(M)_{\text{odd}})$, and we obtain the direct sum decomposition

$$\mathrm{Cl}(q) = \mathrm{Cl}(q)_+ \oplus \mathrm{Cl}(q)_-, \tag{20}$$

where $\mathrm{Cl}(q)_+$ (resp. $\mathrm{Cl}(q)_-$) is generated by sums of even (resp. odd) number of elements of $\pi_q(M)$.

Notice that the product of any two even elements is even, the product of any two elements one of which is odd and one even, is odd, and the product of any two odd elements is even. I short:

$$\begin{aligned} \mathrm{Cl}(q)_+\mathrm{Cl}(q)_+ \subset \mathrm{Cl}(q)_+,\ \mathrm{Cl}(q)_+\mathrm{Cl}(q)_- &\subset \mathrm{Cl}(q)_-, \\ \mathrm{Cl}(q)_-\mathrm{Cl}(q)_- &\subset \mathrm{Cl}(q)_+. \end{aligned} \tag{21}$$

Therefore (since also 1 is an even element) $\mathrm{Cl}(q)_+$ is a subalgebra of $\mathrm{Cl}(q)$. It is called *the even Clifford algebra.*

1.2.3 Anti-derivations

We denote by M^* the dual module, that is the module of all linear functions from M to R.

Lemma 1.18 ([6, Lemma 3.2, p. 43],[4, Lemma 1, p. 141])**.** *Let f be an element of M^*. There exists a unique linear mapping i_f from $T(M)$ to $T(M)$ such that*

1. *We have*
$$i_f(1) = 0, \tag{22}$$

2. *For all $x \in M \subset T(M)$, $u \in T(M)$. we have*
$$i_f(x \otimes u) = f(x)u - x \otimes i_f(u). \tag{23}$$

The map $f \mapsto i_f$ from M^* to linear transformations on $T(M)$ is linear. We have

(i) $i_f(T^pM) \subset T^{p-1}M,$

(ii) $i_f^2 = 0,$

(iii) $i_f i_g + i_g i_f = 0$, *for all $f, g \in M^*$.*

If q is a quadratic form on M then the ideal $J(q)$ is stable under i_f, that is $i_f(J(q)) \subset J(q)$, and thus i_f defines the mapping, denoted by $\bar{i}_f$, on the quotient Clifford algebra $\mathrm{Cl}(q) = T(M)/J(q)$*:*

$$\pi_q \circ i_f = \bar{i}_f \circ \pi_q. \tag{24}$$

On $\mathrm{Cl}(q)$ *we then have*

(iv) $\bar{i}_f(1) = 0$, $(1 \in \mathrm{Cl}(q))$

(v) *For all $x \in M$, $w \in \mathrm{Cl}(q)$, we have*
$$\bar{i}_f(\pi_q(x)w) = f(x)w - \pi_q(x)\bar{i}_f(w). \tag{25}$$

■

Corollary 1.19. *[6, Corollary, p. 44] If M is a vector space, then the mapping $\pi_q : M \to \mathrm{Cl}(q)$ is injective and we can identify M with $\pi_q(M)$.*

The proof goes as follows. Let x be a nonzero vector in M and let f be an element from M^* for which $f(x) = 1$ (cf. Section 1.1.1). Let i_f be as in Lemma 1.18. Setting $w = 1$ in Eq. 25 we get $i_f(\pi_q(x)) = f(x)1 \neq 0$, therefore $\pi_q(x) \neq 0$.

1.2.4 Bourbaki's application λ_F

Definition 1.20. *Let F be a bilinear form on M. Then every $x \in M$ determines a linear form f_x on M defined as $f_x(y) = F(x, y)$. We will denote by $\bar{i}^F_x$ the antiderivation $\bar{i}_{f_x}$ described in Lemma 1.18. In particular we have:*

(i) $\bar{i}^F_x(1) = 0$, $(1 \in \mathrm{Cl}(q))$

(ii) *For all $y \in M$, $w \in \mathrm{Cl}(q)$, we have*

$$\bar{i}^F_x(yw) = F(x, y)w - y\,\bar{i}^F_x(w). \tag{26}$$

Proposition 1.21. *With the notation as in the Definition 1.20, for $y_1, ..., y_n$ in $\mathrm{Cl}(q)$ we have*

$$\bar{i}^F_x(y_1...y_n) = \sum_{j=1}^{n}(-1)^{n-1}F(x, y_j)y_1...\hat{y}_j...y_n, \tag{27}$$

where $\hat{y}_j$ means that this entry is omitted in the product.

In particular if $F(x, y_j) = 0$ for all $j = 1, .., n$, then $\bar{i}^F_x(y_1...y_n) = 0$.

Proof. The proof follows immediately from the definition by induction. □

The following Lemma is taken from Bourbaki [4, p. 142-143]. As we will see it has far reaching consequences.

Lemma 1.22. *There exists a unique linear mapping $\lambda_F : T(M) \to T(M)$ such that*

$$\lambda_F(1) = 1, \tag{28}$$

$$\lambda_F(x \otimes u) = i^F_x(\lambda_F(u)) + x \otimes \lambda_F(u),\ x \in M. \tag{29}$$

For all $f \in M^*$ we have

$$\lambda_F \circ i_f = i_f \circ \lambda_F. \tag{30}$$

If F and G are two bilinear forms on M, then

$$\lambda_F \circ \lambda_G = \lambda_{F+G}. \tag{31}$$

Foe every bilinear form F the linear mapping $\lambda_F : T(M) \to T(M)$ is a bijection. ■

The consequence of this Lemma for Clifford algebras is described in the following Proposition.

Proposition 1.23 ([4, Proposition 3, p. 13]). *Let q and q' be two quadratic forms on M such that $q'(x) = q(x)+F(x,x)$, where $F(x,y)$ is a bilinear form. The mapping λ_F maps the ideal $J(q)$ onto the ideal $J(q)$ and it defines an isomorphism, denoted $\bar{\lambda}_F$ of the R-module $C(q')$ onto the R-module* $\mathrm{Cl}(q)$:

$$\pi_q \circ \lambda_F = \bar{\lambda}_F \circ \pi_{q'}. \tag{32}$$

Note 1.24. In the following we will always assume that M is a vector space. *Therefore, in particular, M can be identified with $\pi_q(M) \subset \mathrm{Cl}(q)$.*

Proposition 1.25. *For all $x \in M$, $w \in \mathrm{Cl}(q)$ we have*

$$\begin{aligned} \bar{\lambda}_F(1) &= 1, \\ \bar{\lambda}_F(x) &= x, \\ \bar{\lambda}_F(xw) &= \bar{i}^F_x(\bar{\lambda}_F(w)) + x\bar{\lambda}_F(w). \end{aligned} \tag{33}$$

If F, G are bilinear forms, if $q''(x) = q'(x) + G(x,x)$ and $q'(x) = q(x) + F(x,x)$, then

$$\bar{\lambda}_{F+G} = \bar{\lambda}_F \circ \bar{\lambda}_c G. \tag{34}$$

Since $\bar{\lambda}_0$ is the identity map, we thus have

$$(\bar{\lambda}_F)^{-1} = \lambda_{(-F)}. \tag{35}$$

Proof. The only property in the two propositions above that is not taken directly from Ref. [4] is the formula (33). But it follows immediately by applying π_q to both sides of Eq. (29) and making use of Eqs. (25) and (32). Eq. (34) follows directly from Eq. (31) and the definition of the quotient mappings, as we have the general property that *composition of two quotient mappings is the quotient of their composition.* ■

Note 1.26. *Notice that in Eq. (33) the multiplication xw on the left is in the algebra $C(q')$, while the multiplication $x\bar{\lambda}_F(w)$ on the right is in the algebra* $\mathrm{Cl}(q)$.

Lemma 1.27. *Let M be a vector space. If $x_1, ..., x_n$ are in M and if $F(x_i, x_j) = 0$ for $i < j$, then*

$$\bar{\lambda}_F(x_1...x_n) = x_1...x_n. \tag{36}$$

Proof. The proof is by induction. For $n = 1$ the statement evidently holds. Let us assume it holds for n and suppose we add x such that $(x, x)i) = 0$ for $i = 1, ..., n$. Then, using Eq. (33) we have

$$\bar{\lambda}_F(xx_1...x_n) = \bar{i}_x^F(x_1...x_n) + xx_1...x_n.$$

But then, using Proposition 1.21, we get $\bar{i}_x^F(x_1...x_n) = 0$. $\square$

The following immediate corollary can be found in Bourbaki [2, Exercise 3c, p. 154]

Corollary 1.28. *Let M be a vector space over a field of characteristic $\neq 2$, q a quadratic form, and Φ the associated bilinear form. Let $F(x, y) = \frac{1}{2}\Phi(x, y)$, so that $q(x) = F(x, x)$, and denote $\mu_q = \bar{\lambda}_F$, so that $\mu_q : \mathrm{Cl}(q) \to \Lambda(M)$. If $x_1, ..., x_n$ are in M and if they are pairwise orthogonal, i.e. $F(x_i, x_j) = 0$ for $i \neq j$ then*

$$\mu_q(x_1...x_n) = x_1 \wedge .... \wedge x_n. \tag{37}$$

1.2.4.1 The mapping λ_F as an exponential Here we assume that the ring R is the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. Replacing F by tF and G by sF, $t, s \in \mathbb{R}$ from the property (31 we obtain

$$\lambda_{tF} \circ \lambda_{sF} = \lambda_{(t+s)F}. \tag{38}$$

Since $\lambda_0 = Id$, it follows that there exists a linear operator a_F on $T(M)$ such $\lambda_{tB} = \exp(ta_F)$. We can the properties defining a_F by replacing F by tF in Eqs. (28) and (29) defining λ_F, and differentiating with respect to t at $t = 0$. We notice that $\lambda_{tF}(x) = x$ and that, since λ_F is linear in F, we have $i_x^{tF} = ti_x^F$. Taking all this into account we obtain:

$$\begin{aligned} a_F(1) &= 0, & (39)\\ a_F(x) &= 0,\ x \in M, & (40)\\ a_F(x \otimes u) &= i_x^F(u) + x \otimes a_F(u). & (41) \end{aligned}$$

In particular we get

$$\begin{aligned} a_F\,(x \otimes y) &= F(x, y) & (42)\\ a_F\,(x \otimes y \otimes z) &= F(x, y)z - F(x, z)y + F(y, z)x, & (43) \end{aligned}$$

$$\begin{aligned} a_F\,(x \otimes y \otimes z \otimes u) = F(x, y)z \otimes u - F(x, z)y \otimes u + F(y, z)x \otimes u +\\ F(z, u)x \otimes y - F(y, u)x \otimes z + F(x, u)y \otimes z. \end{aligned} \tag{44}$$

Note 1.29. *There is an important particular case when the bilinear form F is antisymmetric: $F(x,y) = -F(y,x)$. In this case $q'(x) = q(x) + F(x,x) = q(x)$. Therefore $\bar{\lambda}_F$ maps every $\mathrm{Cl}(q)$ into itself. In particular it maps into itself the exterior algebra $\Lambda(M)$ of M. Thus we can rewrite the equations (42)-(44) replacing $\otimes$ by $\wedge$.*

In quantum physics exterior algebra is used to describe the Fock space of a Fermi field. The operator a_F removes two particles from a multiparticle state - it acts like a annihilation of a pair operator. Pairs of Fermions seem to be of some importance in theories of superconductivity. Thus it may be speculated that operators similar to λ_F and a_F may be relevant for mathematical models of physical phenomena similar to superconductivity.

1.3 Graded structure of a Clifford algebra

Here we assume that M **is a finite dimensional vector space over a field with characteristic** $\neq 2$.

Remark 1.30. *If $e_1, ..., e_n$ is a basis in M, then the tensor algebra $T(M)$ has the basis $1, e_i, e_{i_1} \otimes e_{i_2}, ..., e_{i_1} \otimes ... \otimes e_{i_p}, ...$ Thus a general element of the tensor algebra can be represented as a finite sequence of tables $t, t^i, t^{i_1 i_2}, ..., t^{i_1 \dots i_p}$ where $t, t^i, t^{i_1 \dots i_p}$ (with $i, i_1, ...i_p = 1, ..., n$) are scalars. In the Clifford algebra we skip the symbol of tensor multiplication and we restrict ourselves to $i_1 < ... < i_p$, with $p \leq n$. The tensor algebra is always infinite dimensional, the Clifford algebra is always of the dimension 2^n.*

We will be using the notation as in Corollary 1.28. In particular $\Lambda(M)$ is the exterior algebra over M, q is a quadratic form, F is the unique symmetric bilinear form such $q(x) = F(x,x)$, and $\mu_q = \bar{\lambda}_F$ is the vector space isomorphism $\mu_q : \mathrm{Cl}(q) \to \Lambda(M)$ with the properties that $\mu_q(1) = 1$, $\mu_q(x) = x$ for $x \in M$, and

$$\mu_q(xw) = i_x^F(\mu_q(w)) + w \wedge \mu_q(w). \tag{45}$$

In particular if $x_1, ..., x_n$ are pairwise orthogonal, i.e $F(x_i, x_j) = 0$ for $i \neq j$, then

$$\mu_q(x_1 ... x_n) = x_1 \wedge ... \wedge x_n. \tag{46}$$

From Proposition 1.13 we know that M admits an orthogonal basis $\{e_i\}$,$i = 1, ..., n$. We choose this basis, then μ_q maps each product $e_{i_1} ... e_{i_p}$ in $\mathrm{Cl}(q)$ to the product $e_{i_1} \wedge ... \wedge e_{i_p}$ in the exterior algebra $\Lambda(M)$. The exterior algebra $\Lambda(M)$ is Z-graded:

$$\Lambda(M) = \bigoplus_{p=0}^{n} \Lambda^p(M), \tag{47}$$

where $\Lambda^p(M)$ is $\binom{n}{p} = \frac{n!}{p!(n-p)!}$ dimensional, and the elements $e_{i_1} \wedge ... \wedge e_{i_p}$ (p-vectors) with $i_1 < ... < i_p$ form a basis in $\Lambda^p(M)$. For $p > n$ we have $\Lambda^p(M) = \{0\}$, for $p = 1$ we have $\Lambda^1(M) = M$, and for $p = n$ we have that $\Lambda^n(M)$ is 1-dimensional, spanned by $e_1 \wedge ... \wedge e_n$. For $p = 0$ we have that $\Lambda^0(M)$ is the basic field. The whole exterior algebra $\Lambda(M)$ is $\sum_{p=0}^{n} \binom{n}{p} = 2^n$ dimensional.

We can use the linear isomorphism μ_q to transfer the graded structure of the exterior algebra back to $\mathrm{Cl}(q)$ by defining

$$C^p(q) = \mu_q^{-1}(\Lambda^p(M)). \tag{48}$$

The subspaces $C^p(q)$ are $\binom{n}{p}$-dimensional. Moreover,if $e_1, ..., e_n$ is *any* orthogonal basis for M, then the products $e_{i_1}...e_{i_p}$, $(i_1 < i_2 < ... < i_p)$, form a basis $C^p(q)$.

Remark 1.31. *One has to be careful here. While it is true that any set of linearly independent vectors can be extended to a basis, it is not true, in general, that any set of mutually orthogonal vectors can be extended to an orthogonal basis. As a simple counterexample we can take two-dimensional space* $\mathbb{R}^2$ *with quadratic form* $q(x_1, x_2) = x_1^2 - x_2^2$*, and the bilinear form* $F(x, y) = x_1y_1 - x_2y_2$*. The vector* e_1 *with components* $(1, 1)$ *has the property* $q(e_1) = 0$*, but any vector orthogonal to this vector is automatically proportional to* e_1*. Thus* e_1 *can not be extended to an orthogonal basis.*

1.3.1 The center $Z(q)$ of $\mathrm{Cl}(q)$

For any algebra A its center $Z(A)$ is defined as the set of all these elements of the algebra that commute with every element of A

$$Z(A) = \{u \in A : ua = au \text{ for all } a \in A\}. \tag{49}$$

It follows from the definition that the center of any algebra A is a subalgebra of A, and that it always contains the scalar multiples of the identity of A. With the assumptions and notation as in Sec. 1.3 we will now find the center of the Clifford algebra $\mathrm{Cl}(q)$. First we will do it for a general, possibly degenerate q, then we will specialize to the case of nondegenerate q. Instead of stating the result first, and the providing a proof, we will take the opposite way: first we will discuss the subject and derive the result, and only then make it precise in the form of a proposition. We will use the fact that the algebra $\mathrm{Cl}(q)$ is graded into even and odd parts, cf. Eq. (21).

Suppose u is an element of the center and let us split it into the even and odd parts

$$u = u_0 + u_1,\ u_0 \in \mathrm{Cl}(q)_+, u_1 \in \mathrm{Cl}(q)_-. \tag{50}$$

Since u commutes with all elements of the algebra, it commutes, in particular, with all even elements $a_0 \in \mathrm{Cl}(q)_+$

$$(u_0 + u_1)a_0 = a_0(u_0 + u_1)$$

or

$$u_0a_0 - a_0u_0 = a_0u_1 - u_1a_0.$$

On the left we have even element, on the right - odd. Therefore both must be zero. Thus $u_0a_0 = a_0u_0$ and $a_0u_1 = u_1a_0$. We can do the same for odd elements a_1. The result is that if $u = u_0 + u_1$ is in the center, then the even part U_0 and the odd part u_1 are in the center. Therefore we can look separately for even and for odd elements of the center.

Let us first look for even elements u_0 in the center. We choose an orthogonal basis e_i in M, and the corresponding basis $e_{i_1}...e_{i_p}$, $(i_1 < i_2 < ... < i_p)$, in $C^p(q)$. For u to commute with all the elements of the algebra is the same as to commute with all elements of the basis $e_{i_1}...e_{i_p}$. We can also write u_0 as a linear combination of even elements, $e_{i_1}...e_{i_p}$, p even, of the basis of the algebra. Let us select the first vector e_1 of the basis e_i. We can then split u_0 into the part v_0 that is the linear combination of those $e_{i_1}...e_{i_p}$ that does not contain e_1, and the second part, made of those $e_{i_1}...e_{i_p}$ that contain e_1. Which we write as follows:

$$u_0 = v_0 + e_1v_1, \tag{51}$$

where v_0 is even and does not contain e_1, and v_1 is odd and does not contain e_1. But now u_0 must commute with e_1, which means

$$e_1(v_0 + e_1v_1) = (v_0 + e_1v_1)e1. \tag{52}$$

Since v_1 is odd, and since it does not contain e_1, it follows that v_1 anticommutes with e_1, i.e. $v_1e_1 = -e_1v_1$. Since v_0 is even and it does not contain v_1, it commutes with e_1. Therefore, from Eq. (52 we get that $e_1^2v_1 = 0$. If $e_1^2 \neq 0$, which certainly happens if q is nondegenerate, we deduce that $v_1 = 0$. Therefore u_0 is even and does not contain e_1. The same we can repeat with e_2. We can move to the front in the expression e_2v_1 changing the sign of v_1. The result is that u_0 does not contain in its expansion any element e_i with $e_i^2 \neq 0$.

We now investigate odd elements in the center. As before we write

$$u_1 = v_1 + e_1v_0,$$

where v_1 is odd, v_0 is even, and neither v_1 nor v_0 does not contain e_1. This time e_1 commutes with v_0, therefore all we get from $u_1e_1 = e_1u_1$ is that v_1

commutes with e_1, which implies that $v_1 = 0$. Repeating this reasoning for e_2, e_3, etc. we conclude that u_1 is the proportional to the product $e_1....e_n$ of all basis elements. Since u_1 is odd, this can happen only if n is odd.

We summarise the above in the proposition below:

Proposition 1.32. *The even part of the center of* $\mathrm{Cl}(q)$ *consists of linear combinations of the even products of basis elements of* M *whose square is zero, and of the identity. The odd part of the center consists of the scalar multiples of the element* $e_1....e_n$ *if the dimension of* M *is odd, and consists of zero alone if the dimension of* M *is even.*

1.3.2 The algebras $\mathrm{Cl}_{p,q,r}$ in the real case (c.f. [15])

Let us now concentrate on the real case, when M is a real vector space of dimension n. In that case if e_i is an orthogonal basis, the squares e_i^2 are the real numbers. If e_i^2 i positive, we will redefine e_i replacing it with $e_i \mapsto e_i/\sqrt{e_i^2}$. the new e_i has now square $+1$. If e_i^2 is negative, we replace $e_i \mapsto e_i/\sqrt{-e_i^2}$, and the new e_i^2 is now -1. In this way we diagonalize the quadratic form q, so that on our basis vectors it has only values $+1$, -1, or 0. We now reorganiza our basis so, that we have first basis vectors with square $+1$, say there are p of them, $e_1, ..., e_p$, then we have q basis vectors with square -1, $e_{p+1}, ..., e_{p+q}$, finally we have r basis vectors with square zero, $e_{p+q+1}, ..., e_{P+q+r}$, with $p+q+r = n$. We call such a basis *orthonormal.* The corresponding Clifford algebra is then denoted as $\mathrm{Cl}_{p,q,r}$. If $r = 0$, we simply write $\mathrm{Cl}_{p,q}$, and when $r = 0$ and $q = 0$, we write Cl_p.

We will use the notation $\mathrm{Cl}^0_{p,q,r}$ for the even subalgebra of $\mathrm{Cl}_{p,q,r}$.

We will now demonstrate several simple isomorphisms between Clifford algebras for different p and q.

Lemma 1.33. *For* $p \geq 1$ *we have the isomorphisms of algebras* $\mathrm{Cl}_{p,q,r} \simeq \mathrm{Cl}_{q+1,p-1,r}$.

Proof. Indeed, let $e_1, ..., e_p, e_{p+1}, ..., e_{p+q}, e_{p+q+1}, ..., e_{p+q+r}$ be an orthonormal basis for $\mathrm{Cl}_{p,q,r}$. we define a new basis $\tilde{e}_i$ as follows:

$$\tilde{e}_i = \begin{cases} e_1,\ i = 1; \\ e_i e_1,\ i = 2, ..., p+q+r. \end{cases} \tag{53}$$

We find that all $\tilde{e}_i$ anticommute with each other, that $\tilde{e}_1^2 = 1$, $\tilde{e}_2^2 = ... = \tilde{e}_p^2 = -1$, $\tilde{e}_{p+1}^2 = ... = \tilde{e}_{p+q}^2 = 1$, and $\tilde{e}_{p+q+1}^2 = ... = \tilde{e}_{p+q+r}^2 = 0$. Therefore the basis $\tilde{e}_i$ generates the Clifford algebra $\mathrm{Cl}_{q+1,p-1,r}$. Yet this is the same algebra as the original one. □

Remark 1.34. *The above isomorphism is the isomorphism of two algebras. That means there is a bijective linear map* $\phi : \mathrm{Cl}_{p,q,r} \to \mathrm{Cl}_{q+1,p-1,r}$ *such that* $\phi(ab) = \phi(a)\phi(b)$ *for all* $a, b \in \mathrm{Cl}_{p,q,r}$. *The two algebras are isomorphic as abstract algebras, identity is mapped into identity, but, for instance, their graded structures are not isomorphic. The mapping* ϕ *does not map odd elements into odd elements, also the main automorphism and anti-automorphism are different for the two algebras.*

Lemma 1.35. *For* $p \geq 4$ *we have the isomorphism of algebras* $\mathrm{Cl}_{p,q,r} \simeq \mathrm{Cl}_{p-4,q+4,r}$.

Proof. With the notation as in the proof of Lemma 1.33 we set

$$\tilde{e}_i = \begin{cases} e_i e_1 e_2 e_3 e_4, \ i = 1, 2, 3, 4; \\ e_i, \ i = 5, ..., n. \end{cases} \tag{54}$$

Notice that e_i anticommutes with $e_1 e_2 e_3 e_4$ for $i = 1, ..., 4$. Therefore, for $i \leq 4$, we have

$$\tilde{e}_i^2 = -(e_1 e_2 e_3 e_4)^2 = -1.$$

Therefore the first four vectors of the basis change it squares from $+1$ to -1. □

Remark 1.36. *In this case both algebras have the same even and odd parts. Therefore main automorphisms are the same. But main anti-automorphisms are not the same. Calculating the main anti-automorphism of the first algebra on* $\tilde{e}_1$ *we find*

$$\tilde{e}_1^\tau = (e_2 e_3 e_4)^\tau = e_4 e_3 e_2 = e_2 e_4 e_3 = -e_2 e_3 e_4 = -\tilde{e}_1$$

while the main anti-automorphism of the second algebra should leave $\tilde{e}_1$ *unchanged.*

Notice that in both lemmas the new generators $\tilde{e}_i$ *are linearly independent as they are proportional to the elements of the standard basis (without counting the identity* 1*)* $e_{i_1}...e_{i_k}$, $i_1 < .... < i_k$, $k = 1...n$ *of the Clifford algebra .*

Lemma 1.37. *For* $q \geq 1$ *we have the isomorphism*

$$\mathrm{Cl}^0_{p,q,r} \simeq \mathrm{Cl}_{p,q-1,r}, \tag{55}$$

where $\mathrm{Cl}^0_{p,q,r}$ *denotes the even subalgebra of* $\mathrm{Cl}_{p,q,r}$.

Proof. Let e_i be an orthonormal basis with $e_i^2 = 1$, for $i = 1, ..., p$, $e_i^2 = -1$ for $i = p+1, ..., p+q$, and $e_i^2 = 0$ for $i = p+q+1, ..., n$. We define $\tilde{e}_i$, for $i = 1, ..., e_n$ as

$$\tilde{e_i} = e_i e_{p+q}. \tag{56}$$

We may skip $i = p+q$, since then $\tilde{e}_i = -1$. We obtain this way $n-1$ mutually anticommuting elements, with p squares $+1$, $q-1$ squares -1 and r squares zero. Therefore they generate the algebra $\mathrm{Cl}_{p,q-1,r}$. On the other hand they are all even elements of $\mathrm{Cl}_{p,q,r}$, and every even element of $\mathrm{Cl}_{p,q,r}$ can be obtained using $\tilde{e}_i$. Thus the lemma holds. □

1.3.2.1 Examples in low dimensions

It is important to know an explicit form of Clifford algebras in low dimensions, since then we can you the periodicity properties for finding their forms in higher dimensions.

1.3.2.1.1 $\mathrm{Cl}_0 \simeq \mathbb{R}$. Here $n = 0$, so the Clifford algebra is $2^0 = 1$-dimensional. The vector space is in this case zero-dimensional, it consists of the vector 0 alone. The Clifford algebra consists just of the scalar multiples of the identity.

1.3.2.1.2 $\mathrm{Cl}_1 \simeq \mathbb{R} \oplus \mathbb{R}$. Here $n = 1$ and the Clifford algebra is 2-dimensional. Apart of the identity there is just one basis vector with square 1. Sometimes it is convenient to represent such a direct sum as diagonal matrices, in this case with real numbers on the diagonal. We can choose:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{57}$$

1.3.2.1.3 $\mathrm{Cl}_{0,1} \simeq \mathbb{C}$. Apart of the identity we have one generator (basis vector) with square minus one $e_1^2 = -1$. We can represent the generators by the matrices:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{58}$$

Identifying e_1 with i, the imaginary square root of -1, the algebra becomes isomorphic to complex numbers $\mathbb{C}$.

1.3.2.1.4 $\mathrm{Cl}_{0,0,1}$ - the Dual numbers Here we have one generator (basis vector) with square 0. It can be represented by the matrix

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{59}$$

The algebra is known under the name of *dual numbers*.

1.3.2.1.5 $\mathrm{Cl}_{0,2} \simeq \mathbb{H}$**.** The quadratic form[3] $q(x)$ is in this case $Q(x^1, x^2) = -(x^1)^2 - (x^2)^2$. We have two anticommuting generators e_1, e_2 with squares -1

$$e_1^2 = e_2^2 = -1,\; e_1 e_2 = -e_2 e_1. \tag{60}$$

The elements $1, e_1, e_2, e_{12} = e_1 e_2$ form the basis of the algebra. We find that e_{12} has also square -1 and it anticommutes with e_1 and e_2. Using the substitution

$$e_1 \to \mathrm{i}, \qquad e_2 \to \mathrm{j}, \qquad e_{12} \to \mathrm{k}$$

While the algebra of quaternions is isomorphic to the Clifford algebra $\mathrm{Cl}_{0,2}$ the isomorphism is not a *natural* one. The natural function of quaternions is to implement rotations in $\mathbb{R}^3$. In order to understand better the role of quaternions as representing elements of $\mathrm{Cl}_{0,2}$ let us find how are the Clifford algebra operations such as trace, main involution and main anti-involution represented in $\mathbb{H}$.

Every quaternion q is written as $q = \alpha_0 + \alpha_1 \mathrm{i} + \alpha_2 \mathrm{j} + \alpha_3 \mathrm{k}$, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are real numbers. This corresponds to the element q of the Clifford algebra $\mathrm{Cl}_{0,2}$

$$q = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_{12} e_{12}. \tag{61}$$

Therefore the trace of q, as it is defined in Sec. 1.3.3, is simply the scalar part α_0 of the quaternion. The main involution α is an automorphism of the algebra that changes the signs of odd vectors. In our case it should change the sign of i and j, but not of k. It is easy to guess its form acting on quaternions[4]:

$$\alpha(q) = \mathrm{k} q \mathrm{k}^{-1}. \tag{62}$$

Anti-automorphism τ should change the order of multiplication, but should not change the signs of e_1 ande_2. Quaternions have a well known anti-automorphism, the conjugation $q \mapsto q^*$ which changes the sign of the imaginary units $\mathrm{i}, \mathrm{j}, \mathrm{k}$. To make it not to change the signs of e_1, e_2 we must combine it with the previous automorphism. Thus:

$$\tau(q) = \mathrm{k}\, q^*\, \mathrm{k}^{-1}. \tag{63}$$

[3] The symbol q for the quadratic form should not be confused with the symbol q used for a generic quaternion.

[4] Where of course $\mathrm{k}^{-1} = -\mathrm{k}$.

1.3.2.1.6 $\mathrm{Cl}_3 \cong \mathrm{Mat}(2,\mathbb{C})$. The algebra is $2^3 = 8$ dimensional. We have three anticommuting generators with squares 1. They can be represented by the Pauli matrices:

$$1 \mapsto \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \mapsto \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$e_2 \mapsto \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad e_3 \mapsto \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have:

$$\sigma_1\sigma_2 = \mathrm{i}\sigma_3, \quad \sigma_2\sigma_3 = \mathrm{i}\sigma_1, \quad \sigma_3\sigma_1 = \mathrm{i}\sigma_2, \quad \sigma_1\sigma_2\sigma_3 = \mathrm{i}.$$

Notice that while

$$\sigma_1\sigma_2$$

is proportional to σ_3 with the complex proportionality constant, it is independent of σ_3 as a element of the real vector space. The eight complex matrices $\sigma_0, \sigma_i, \sigma_{ij}\ (i < j)$, and $\sigma_1\sigma_2\sigma_3 = \mathrm{i}$ form a real basis in the space of 2×2 of complex matrices. Every complex matrix 2×2 can be written as a linear combination of these eight matrices with real coefficients. The space of 2×2 complex matrices has 4 complex dimensions, that is 8 real dimensions.

As we did it with quaternions, so here we will identify the trace, the main automorphism, and the main anti-automorphism of the Clifford algebra Cl_3 realized as the algebra of all complex 2×2 matrices.

The trace is easy, it should be the real coefficient in front of the identity matrix. So it must be 1/2 of the real part of the ordinary trace of the matrix.

We now consider the main automorphism. It should change the sign of the three Pauli matrices. Matrices σ_1 and σ_3 are real, while σ_2 is imaginary. The formula that works can be obtained after some little work. For a complex 2×2 matrix a we find that[5]

$$\alpha(a) = \sigma_2 \bar{a} \sigma_2^{-1}, \tag{64}$$

where $\bar{a}$ denotes the complex conjugated matrix.

The main anti-automorphism should reverse the order of multiplication, but must leave the Pauli matrices unchanged. All three Pauli matrices are Hermitian, therefore the Hermitian conjugate of the complex matrices (complex conjugate transpose, $a \mapsto a^* = \bar{a}^t$) does the job:

$$\tau(a) = a^*. \tag{65}$$

where i, j, and k are imaginary units of quaternions, we obtain the isomorphism $\mathrm{Cl}_{0,2} \simeq \mathbb{H}$ - the algebra of quaternions.

[5]Where, of course, $\sigma_2^{-1} = \sigma_2$.

1.3.2.1.7 $\mathrm{Cl}_{1,1} \simeq \mathrm{Mat}(2,\mathbb{R})$**.** The quadratic form is now $q(x^1, x^2) = (x^1)^2 - (x^2)^2$. The Clifford algebra is $2^2 = 4$-dimensional. We can represented it as the algebra of all real 2×2 matrices (which is also $2\times 2 = 4$-dimensional by defining generators e_1, e_2 with squares 1 and -1 as follows:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{66}$$

The element e_1e_2 is now represented by the matrix

$$e_1e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{67}$$

These four matrices span the whole algebra of 2×2 real matrices. The trace in the Clifford algebra is now 1/2 of the trace of the matrix. The main automorphism is realized as $a \mapsto e_{12} a e_{12}^{-1}$, the main anti-automorphism as $a \mapsto e_1 a^t e_1^{-1}$.

1.3.2.1.8 $\mathrm{Cl}_{p+1,q+1} \simeq \mathrm{Mat}(2, \mathrm{Cl}_{p,q})$. In general the notation $\mathrm{Mat}(2, A)$ denotes the algebra of 2×2 matrices the entries of which are elements of the algebra A. Let e_i be the basis of the $n = p+q$ dimensional space generating the 2^n-dimensional Clifford algebra Cl_{p+q}. Let $\mathbf{1}$ be the identity element of this algebra. We define $n+2$ generators of the algebra $\mathrm{Cl}_{p+1,q+1}$ as follows

$$e_i \mapsto \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix},\ e_+ \mapsto \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},\ e_- \mapsto \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{68}$$

The matrices on the right are 2×2 block matrices, with blocks of the size $2^n \times 2^n$. They form $4 \times 2^n = 2^{n+2}$ algebra.

Of course the isomorphism $\mathrm{Cl}_{1,1} \simeq \mathrm{Mat}(2,\mathbb{R})$ is a particular case of the above general isomorphism, for $n = 0$.

1.3.2.2 The table of real Clifford algebras $\mathrm{Cl}_{p,q}$

The following classification of all real Clifford algebras $\mathrm{Cl}_{p,q}$ can be obtained by following the reasoning like those above (c.f. [15] and references therein):

Theorem 1.38 (Cartan 1908)**.** *We have the following isomorphism of algebras*

$$\mathrm{Cl}_{p,q} \cong \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{R}), & \textit{if } p-q \equiv 0;2 \mod 8 \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{R}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{R}), & \textit{if } p-q \equiv 1 \mod 8 \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}), & \textit{if } p-q \equiv 3;7 \mod 8 \\ \mathrm{Mat}(2^{\frac{n-2}{2}}, \mathbb{H}), & \textit{if } p-q \equiv 4;6 \mod 8 \\ \mathrm{Mat}(2^{\frac{n-3}{2}}, \mathbb{H}) \oplus \mathrm{Mat}(2^{\frac{n-3}{2}}, \mathbb{H}), & \textit{if } p-q \equiv 5 \mod 8. \end{cases}$$

The form of the matrix representation of the algebra shows a specific periodicity with respect to $d = p - q \mod 8$. Table 1 shows all Clifford algebras $\mathrm{Cl}_{p,q}$ for $n = p + q$ from 0 to 12. We notice that for n even they are always isomorphic to full matrix algebras with entries being real, complex or quaternionic. In each case they are being considered as real algebras, so that a complex number is considered to be a pair of real numbers, and a quaternion is considered to be four real numbers.

The important element of each Clifford algebra $\mathrm{Cl}_{p,q}$ is its *volume element*, let us denote it as ω. If e_i is an orthonormal basis, then

$$\omega = e_1 e_2 ... e_n. \tag{69}$$

For n even the volume element anticommutes with all basis vectors e_i. For n odd it always commutes - we know that it spans the center of the algebra (cf. Sec. 1.3.1). In that case it is very important whether its square is $+1$ or -1. Lets us calculate $\omega^2 = e_1 ... e_n \, e_1 ... e_n$. We have to commute e_1 that occurs after e_n through e_n,..,e_2, until we get $(e_1)^2$ at the beginning. Each time we change the sign, because $e_i e_j = -e_j e_i$ for $i \neq j$. Thus we will change the sign $n-1$ times. Then we have to do the same with e_2. This will change the sign $n-2$ times. And so on, until we get $(e_1)^2....(e_n)^2$. Altogether we will change the sign $(n-1) + (n-2)+,,,+1 = n(n-1)/2$ times. On the other hand $e_1^2....e_n^2 = (+1)^p(-1)^q = (-1)^q$. Therefore we obtain:

$$\omega^2 = (-1)^{n(n-1)/2+q}. \tag{70}$$

Now, we have $p + q = n$, $p - q = d$, therefore $(n(n-1)/2 + q = (n^2 - d)/2$, and so

$$\omega^2 = (-1)^{\frac{1}{2}(n^2-d)}. \tag{71}$$

If n is odd then $n = 2k + 1$,, therefore $n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$, therefore $(-1)^{n^2} = -1$ and anos os

$$\omega^2 = (-1)^{\frac{d+1}{2}}. \tag{72}$$

If n is odd, then also d is odd. It is clear from the last formula that ω^2 retains the sign when d increases by 4. When $n = 1 \mod 4$ we have $\omega^2 = -1$, when $n = 3 \mod 4$, we have $\omega^2 = 1$. We have thus showed that the following property holds:

Proposition 1.39. *For n odd we have that*

$$\omega^2 \doteq (e_1 ... e_n)^2 = \begin{cases} 1, & \textit{if } p - q = 1 \mod 4 \\ -1, & \textit{if } p - q = 3 \mod 4 \end{cases} \tag{73}$$

1.3.2.2.1 The case of $p+q$ odd and $p-q=1 \mod 4$ (cf. [11, p. 22]) This is the case when $\omega = e_1 ... e_n$ commutes with all the elements of the algebra. Since ω is odd we have $\alpha(\omega) = -\omega$, where α is the main automorphism (involution) of the algebra (cf. Sec. 1.2.2). Let us introduce π^+, π^- as follows:

$$\pi^{\pm} = \frac{1}{2}(1 \pm \omega). \tag{74}$$

Then $\pi^{\pm}$ are idempotents with sum equal 1:

$$(\pi^{\pm})^2 = \pi^{\pm}, \quad \pi^+ + \pi^- = 1, \quad \pi^+\pi^- = \pi^-\pi^+ = 0. \tag{75}$$

Moreover they commute with every element of the algebra, and we have

$$\alpha(\pi^{\pm}) = \pi^{\mp}. \tag{76}$$

Therefore each element a can be split into two parts $a = \pi^+ a + \pi^- a$, and the whole algebra can be split into two ideals

$$\mathrm{Cl}_{p,q} = \mathrm{Cl}^+_{p,q} \oplus \mathrm{Cl}^-_{p,q}, \tag{77}$$

where

$$\mathrm{Cl}^{\pm}_{p,q} = \pi^{\pm}\mathrm{Cl}_{p,q} = \mathrm{Cl}_{p,q}\pi^{\pm} = \pi^{\pm}\mathrm{Cl}_{p,q}\pi^{\pm}. \tag{78}$$

Moreover the two ideals are isomorphic to each other:

$$\alpha(\pi^{\pm}) = \pi^{\mp}. \tag{79}$$

The above reasoning explains why in Table 1, in every row with odd n and $d = 1 \mod 4$ we have entries of the form ${}^2X$, which is a short notation for $X \oplus$X, where X is one of the full matrix algebras.

$n\backslash d$	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
0	·	·	·	·	·	·	·	·	·	·	·	·	$\mathbb{R}$	·	·	·	·	·	·	·	·	·	·	·	·
1	·	·	·	·	·	·	·	·	·	·	·	$\mathbb{C}$	·	$^2\mathbb{R}$	·	·	·	·	·	·	·	·	·	·	·
2	·	·	·	·	·	·	·	·	·	·	$\mathbb{H}$	·	$\mathbb{R}(2)$	·	$\mathbb{R}(2)$	·	·	·	·	·	·	·	·	·	·
3	·	·	·	·	·	·	·	·	·	$^2\mathbb{H}$	·	$\mathbb{C}(2)$	·	$^2\mathbb{R}(2)$	·	$\mathbb{C}(2)$	·	·	·	·	·	·	·	·	·
4	·	·	·	·	·	·	·	·	$\mathbb{H}(2)$	·	$\mathbb{H}(2)$	·	$\mathbb{R}(4)$	·	$\mathbb{R}(4)$	·	$\mathbb{H}(2)$	·	·	·	·	·	·	·	·
5	·	·	·	·	·	·	·	$\mathbb{C}(4)$	·	$^2\mathbb{H}(2)$	·	$\mathbb{C}(4)$	·	$^2\mathbb{R}(4)$	·	$\mathbb{C}(4)$	·	$^2\mathbb{H}(2)$	·	·	·	·	·	·	·
6	·	·	·	·	·	·	$\mathbb{R}(8)$	·	$\mathbb{H}(4)$	·	$\mathbb{H}(4)$	·	$\mathbb{R}(8)$	·	$\mathbb{R}(8)$	·	$\mathbb{H}(4)$	·	$\mathbb{H}(4)$	·	·	·	·	·	·
7	·	·	·	·	·	$^2\mathbb{R}(8)$	·	$\mathbb{C}(8)$	·	$^2\mathbb{H}(4)$	·	$\mathbb{C}(8)$	·	$^2\mathbb{R}(8)$	·	$\mathbb{C}(8)$	·	$^2\mathbb{H}(4)$	·	$\mathbb{C}(8)$	·	·	·	·	·
8	·	·	·	·	$\mathbb{R}(16)$	·	$\mathbb{R}(16)$	·	$\mathbb{H}(8)$	·	$\mathbb{H}(8)$	·	$\mathbb{R}(16)$	·	$\mathbb{R}(16)$	·	$\mathbb{H}(8)$	·	$\mathbb{H}(8)$	·	$\mathbb{R}(16)$	·	·	·	·
9	·	·	·	$^2\mathbb{C}(16)$	·	$^2\mathbb{R}(16)$	·	$\mathbb{C}(16)$	·	$^2\mathbb{H}(8)$	·	$\mathbb{C}(16)$	·	$^2\mathbb{R}(16)$	·	$\mathbb{C}(16)$	·	$^2\mathbb{H}(8)$	·	$\mathbb{C}(16)$	·	$^2\mathbb{R}(16)$	·	·	·
10	·	·	$\mathbb{H}(16)$	·	$^2\mathbb{R}(32)$	·	$\mathbb{R}(32)$	·	$^2\mathbb{H}(16)$	·	$\mathbb{H}(16)$	·	$^2\mathbb{R}(32)$	·	$\mathbb{R}(32)$	·	$^2\mathbb{H}(16)$	·	$\mathbb{H}(16)$	·	$\mathbb{R}(32)$	·	$\mathbb{R}(32)$	·	·
11	·	$^2\mathbb{H}(16)$	·	$\mathbb{C}(32)$	·	$^2\mathbb{R}(32)$	·	$\mathbb{C}(32)$	·	$^2\mathbb{H}(16)$	·	$\mathbb{C}(32)$	·	$^2\mathbb{R}(32)$	·	$\mathbb{C}(32)$	·	$^2\mathbb{H}(16)$	·	$\mathbb{C}(32)$	·	$^2\mathbb{R}(32)$	·	$\mathbb{C}(32)$	·
12	$\mathbb{H}(32)$	·	$\mathbb{H}(32)$	·	$\mathbb{R}(64)$	·	$\mathbb{R}(64)$	·	$\mathbb{H}(32)$	·	$\mathbb{H}(32)$	·	$\mathbb{R}(64)$	·	$\mathbb{R}(64)$	·	$\mathbb{H}(32)$	·	$\mathbb{H}(32)$	·	$\mathbb{R}(64)$	·	$\mathbb{R}(64)$	·	$\mathbb{H}(32)$

Table 1: Isomorhisms between $\mathrm{Cl}_{p,q}$ and matrix algebras. Here $d = p - q$. Here, for instance, $\mathbb{H}(64)$ denotes the algebra $\mathrm{Mat}(64, \mathbb{H})$, and $^2\mathbb{R}(32)$ denotes the direct sum $\mathrm{Mat}(32, \mathbb{R}) \oplus \mathrm{Mat}(32, \mathbb{R})$. The table has a left-right symmetry (as much as possible) with respect to the vertical $d = 1$ line. The even subalgebra is always North-East of a given entry (cf. Lemma 1.37.)

1.3.3 The trace and the bilinear form on $\mathrm{Cl}(q)$

With the assumptions and the notation as above we have the direct sum decomposition

$$\mathrm{Cl}(q)) = \bigoplus_{p=0}^{n} \mathrm{Cl}(q)^p. \tag{80}$$

We denote by Σ the set of all 2^n ordered sequences $i_1 < ... < i_p$, $(0 \leq p \leq n)$, and for each such sequence $I \in \Sigma$ let e_I be the corresponding element $e_I = e_{i_1}...e_{i_p}$ of the basis of $\mathrm{Cl}(q)^p \subset \mathrm{Cl}(q)$. For $p = 0$ we have the empty set, and we take $e_\emptyset = 1 \in \mathrm{Cl}(q)^0$. Now every element a of $\mathrm{Cl}(q)$ can be uniquely written as

$$a = \sum_{I \in \Sigma} a_I e_I. \tag{81}$$

The coefficients a_I depend on the choice of the orthogonal basis e_i - except of the coefficient $a_\emptyset$, the scalar part of a. We denote it $\mathcal{T}(a)$ and call *the trace.* Thus we have defined a linear functional on the Clifford algebra $\mathrm{Cl}(q)$, with values in the basic field.

Definition 1.40. *We denote by $\mathcal{T}$ the linear functional on $\mathrm{Cl}(q)$ assigning to each element $a \in \mathrm{Cl}(q)$ its scalar part $a_\emptyset \in \mathrm{Cl}(q)^0$ in the direct sum decomposition (80).*

In the Proposition below we denote by $a \mapsto a^\tau$ the main anti-involution of $\mathrm{Cl}(q)$ (cf. Sec. 1.2.2). It is characterized by the following properties: $1^\tau = 1$, $x^\tau = x$ for $x \in M$, $(e_{i_1}...e_{i_p})^\tau = e_{i_p}...e_{i_1}$.

Proposition 1.41. *The functional $\mathcal{T}$ has the following properties:*

(i) $\mathcal{T}(1) = 1$,

(ii) $\mathcal{T}(a^\tau) = \mathcal{T}(a)$, $\forall a \in \mathrm{Cl}(q)$,

(iii) $\mathcal{T}(ab) = \mathcal{T}(ba)$, $\forall a, b \in \mathrm{Cl}(q)$,

(iv) $\mathcal{F}(a, b) \stackrel{df}{=} \mathcal{T}(a^\tau b)$ *is a symmetric, bilinear form on* $\mathrm{Cl}(q)$, *that is non-degenerate if F is a non-degenerate form on M. We have* $\mathcal{T}(a) = \mathcal{F}(1, a) = \mathcal{F}(a, 1)$, $\forall a \in \mathrm{Cl}(q)$.

(v) $\mathcal{F}(ab, c) = \mathcal{F}(b, a^\tau c) = \mathcal{F}(a, cb^\tau)$, $\forall a, b, c \in C(Q)$.

Proof. (i) and (ii) follow immediately from the definition. In order to prove (iii) we notice that if e_i is an orthogonal basis in M, e_I, $I = \{i_1 < ... < i_p\}$ is the corresponding basis in $\mathrm{Cl}(q)$, and $a = \sum_I a_I e_I$, $b = \sum b_J e_J$. We notice that e_i and e_j anticommute for $i \neq j$ and that $e_i e_i = F(e_i, e_i)$ are scalars. Therefore $e_I e_J$ is proportional to e_K where K contains the indices that are in I but not in J, or in J but not in I (the symmetric difference of the sets I and J). Therefore $\mathcal{T}(ab) = \mathcal{T}(\sum_I a_I e_I b_J e_J) = \sum_I a_I b_I \mathcal{T}(e_I e_I) = \mathcal{T}(ba)$. That $\mathcal{F}$ is a symmetric bilinear form follows immediately from (ii) and (iii). F is non-degenerate if and only if all $F(e_i, e_i)$ are non zero, and it is immediate that this happens if and only if all $\mathcal{F}(e_I, e_I)$ are non-zero. The remaining statements follow easily from the definitions and the properties proven above. □

2 Clifford algebra on multivectors

We assume, in this section, that M is vector space over reals or complex, not necessarily finite dimensional. Let $F(x, y)$ be a bilinear form (not necessarily symmetric) on M. We have seen in Proposition 1.25 that the mapping $\bar{\lambda}_F$ maps the Clifford algebra $C(q')$ of the quadratic form $q'(x) = q(x) + F(x, x)$ onto the Clifford algebra $\mathrm{Cl}(q)$. It is a vector space isomorphism, with the inverse mapping being $(\bar{\lambda}_F)^{-1} = \bar{\lambda}_{-F} : \mathrm{Cl}(q) \to C(q')$. Let us take the particular case of $q' = 0$ in which case the algebra $C(q')$ becomes identical to the exterior algebra $\Lambda(M)$. Elements of the exterior algebra are called multivectors and the multiplication of multivectors in the exterior algebra is traditionally denoted by the wedge symbol $x \wedge y$. But using the $\bar{\lambda}$ mapping we can also transport back to $\Lambda(M)$ the multiplication from the Clifford algebra $\mathrm{Cl}(q)$. We will now derive the corresponding formula. Let us take $x \in M \subset \Lambda(M)$ and $u \in \Lambda(M)$. Then $\bar{\lambda}_{-F}(x)$ and $\bar{\lambda}_{-F}(u)$ are in $\mathrm{Cl}(q)$. Now we multiply $\bar{\lambda}_{-F}(x)$ and $\bar{\lambda}_{-F}(u)$ in $c(q)$ and transport back their product $\bar{\lambda}{-F}(x)\bar{\lambda}_{-F}(u)$ to $\Lambda(M)$ using $\bar{\lambda}_F$. We obtain the multiplication rule of the Clifford algebra $\mathrm{Cl}(q)$ expressed in terms of multivectors:

$$xu = \bar{\lambda}_F(\bar{\lambda}_{-F}(x)\bar{\lambda}_{-F}(u)). \tag{82}$$

Notice that we identify the vectors of M with their images in $\mathrm{Cl}(q)$, therefore we can take $\bar{\lambda}_{-F}(x) = x$. We can then use Eq. (33):

$$xu = \bar{\lambda}_F(x\bar{\lambda}_{-F}(u)) = \bar{i}^F_x(\bar{\lambda}_F(\bar{\lambda}_{-F}(u))) + x \wedge \bar{\lambda}_F(\bar{\lambda}_{-F}(u)), \tag{83}$$

or

$$\boxed{xu = x \wedge u + \bar{i}^F_x(u).} \tag{84}$$

We recall the action of the antiderivation $\bar{i}^F_x$

(i) For all $x \in M$ we have

$$\boxed{\bar{i}^F_x(1) = 0,\ (1 \in \Lambda(M))}, \tag{85}$$

(ii) For all $x, y \in M \subset \Lambda(M)$, $w \in \Lambda(M)$, we have

$$\boxed{\bar{i}_x^F(y \wedge w) = F(x,y)w - y \wedge \bar{i}_x^F(w).} \tag{86}$$

The bilinear form F above is in general non-symmetric. It can be split as a sum of its symmetric part $F_s(x,y) = F_s(y,x)$ and antisymmetric part $F_a(x,y) = -F_a(y,x)$:

$$\begin{aligned} F(x,y) &= \frac{1}{2}(F(x,y) + F(y,x)) + \frac{1}{2}(F(x,y) - F(y,x)) \\ &= F_s(x,y) + F_a(x,y). \end{aligned} \tag{87}$$

From Eq. (86) we get

$$xy + yx = 2F_s(x,y), \tag{88}$$

$$xy - yx = 2(x \wedge y + F_a(x,y)). \tag{89}$$

In particular Eq. (88) implies $x^2 = F_s(x,x)$. Therefore the multiplication defined in Eq. (84) determines the Clifford algebra $\mathrm{Cl}(q)$ with $q(x) = F_s(x,x)$, and q does not depend at all on the antisymmetric part F_a of F. And yet Eq. (84) defines different multiplications for different antisymmetric parts of F even if the symmetric parts are the same. However, it follows immediately from the universal property of the Clifford algebras that all these algebras corresponding to different antisymmetric parts of F are isomorphic one to another, as they are all Clifford algebras with the same q. Therefore it is somewhat surprising that in Ref. [1] Ablamowicz and Lounesto decided to take the trouble to verify this obvious property using a computer. They wrote

> "We explicitly demonstrate with a help of a computer that Clifford algebra C(B) of a bilinear form B with a non-trivial anti symmetric part A is isomorphic as an associative algebra to the Clifford algebra C(Q) of the quadratic form Q induced by the symmetric part of B." "

Moreover they attribute the formula (84 defining the Clifford multiplication for an arbitrary, possibly degenerate and not necessarily symmetric bilinear form on multivectors to Oziewicz [14] instead of referring to the classical old algebra book of Bourbaki, originally published by Hermann in 1959 [2].

3 Deformations

Here we will expand the method used in the previous section to include more general *deformations* of Clifford algebras. We will start with presenting the facts discussed before from a somewhat more general perspective. We will start assuming that *M is a vector space over the field R of an arbitrary characteristic (thus including characteristic 2).*

3.1 The additive group of bilinear forms Bil(M)

We will deal with three important sets: the set of all bilinear forms $\mathrm{Bil}(M)$, the set of all alternate forms $\mathrm{Alt}(M)$, and the set of all quadratic forms $\mathrm{Quad}(M)$.[6] Each of these sets is, in fact, a vector space. But we will be mainly interested that these sets are *Abelian groups* with respect to the addition "+". We can associate with these sets the following diagram:

$$\mathrm{Alt}(M) \longrightarrow \mathrm{Bil}(M) \xrightarrow{\pi_B} \mathrm{Quad}(M), \tag{90}$$

meaning that $\mathrm{Alt}(M)$ is a subgroup of $\mathrm{Bil}(M)$ and that every quadratic form $q(x)$ can be obtained from some bilinear form $F(x, y)$ via $q(x) = F(x, x)$, with F and F' determining the same $q(x)$ if and only if $F'(x, y) - F(x, y) = A(x, y)$ where $A(x, y)$ is alternate, i.e. $A(x, x) = 0$ for all $x \in M$. This last property has been discussed in Remark 1.12. The mapping π_B associates with every bilinear form F the quadratic form $q(x) = F(x, x)$.

The sequence in Eq.(90) is called *exact*, which means that the map $\mathrm{Alt}(M) \to \mathrm{Bil}(M)$ is injective, and that $\mathrm{Alt}(M)$ is the kernel of the map π_B. What we have can be summarized by saying that we have a *principal bundle* - the group $\mathrm{Bil}(M)$ over the base $\mathrm{Quad}(M)$ - the homogeneous space $\mathrm{Quad}(M) = \mathrm{Bil}(M)/\mathrm{Alt}(M)$, as depicted in Fig. 1.

Remark 3.1. *Here and in the following we are using the language of fiber bundles in an informal way, without paying any attention whatsoever to topology, since topology is not needed in these general algebraic considerations. Topology will come back when we will specify the arbitrary field R to become real or complex numbers.*

Sometimes fiber bundles are graphically represented three dimensionally as in Fig.2. That representation is inappropriate in our case, as it may suggest that each fiber has a distinguished point. But this is not the case in general. While in Remark 1.12 we have indeed constructed a bilinear form from a quadratic form, the construction there was dependent on the choice

[6] I am following the notation used in Ref. [7].

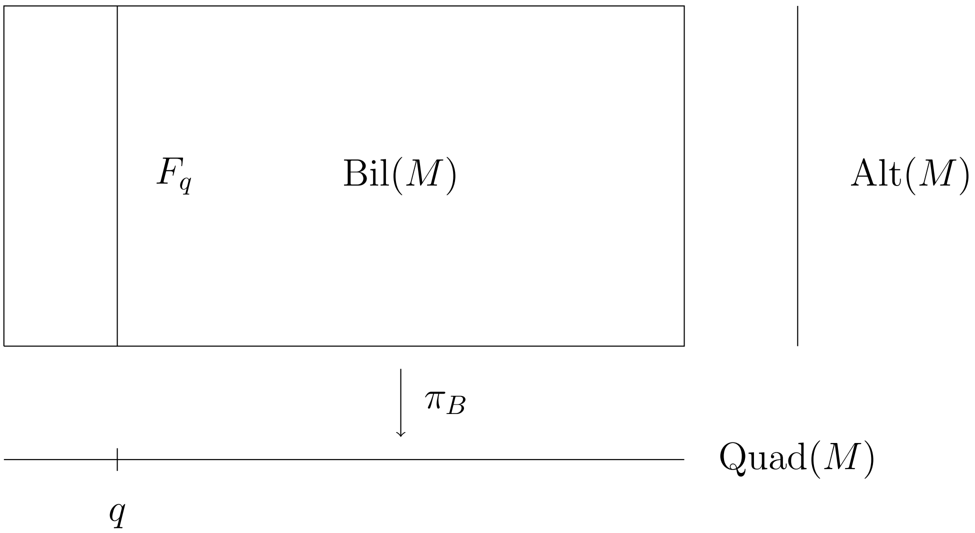

Figure 1: Principal bundle $\mathrm{Bil}(M)$ of bilinear forms over the base $\mathrm{Quad}(M)$ of quadratic forms, with structure group $\mathrm{Alt}(M)$ of alternate forms. The fiber F_q over q consists of all bilinear forms $F(x, y)$ such that $q(x) = F(x, x)$, i.e. $q = \pi_B(F)$.

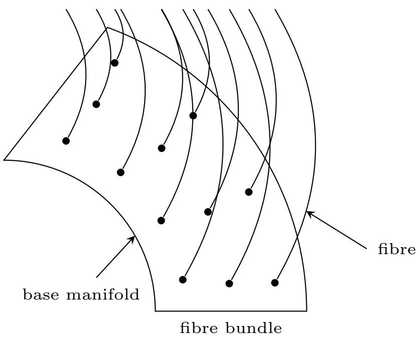

Figure 2: An artistic 3D drawing of a fiber bundle - inappropriate in our context

of a basis in M. Of course there is a distinguish point on each fiber when the field R admits division by 2 - in that case for each quadratic form there is a unique symmetric form in each fibre, namely $F(x,y) = \frac{1}{2}\Phi(x,y)$, where Φ is the bilinear form associated with q.

3.2 The bundle of Clifford algebras

For every quadratic form $q \in \text{Quad}$ we have constructed (see Section 1.2) the Clifford algebra $\text{Cl}(q) = T(M)/J(q)$. We denote by $C(M)$ the collection of all these Clifford algebras:

$$C(M) = \bigcup\{\text{Cl}(q) : q \in \text{Quad}(M)\}. \tag{91}$$

Then to give $C(M)$ the structure of a vector bundle over the base $\text{Quad}(M)$, we need to provide it with local coordinates that enable us represent $\text{Cl}(M)$ as a cartesian product of the base and of a vector space. In fact in our case we can provide not only local but also global coordinates. To this end let $\{e_i\}_{i\in I}$ be a basis in the vector space M, with a well ordered index set I. We the have the following important result (see [2, Theorem 1, p. 145]:

Theorem 3.2. *Assume that $\{e_i\}_{i\in I}$ is a basis in M, with a well ordered index set I. For every finite part H of I let us set $e_H = e_{i_1}\cdots e_{i_n}$ where $\{i_1,\cdots,i_n\}$ is the ordered sequence of all elements of H: $i_1 < \cdots < i_n$. Then the the elements e_H, with H running through all finite subsets of I form a basis for* $\text{Cl}(q)$

Proof. We follow the proof as given in Ref. [2, Theorem 1, p. 145], with only slight adaptations. The proof assumes that we already know that the result holds for the exterior algebra $\Lambda(M) = \text{Cl}(0)$, within which context it is a standard property. Therefore $e_H = e_{i_1}\wedge\cdots\wedge e_{i_n}$ form a basis in $\text{Cl}(0)$. Given now $q \in \text{Quad}(M)$ we construct bilinear form $F(x,y)$ as in Remark 1.12, but this time for the form $-q$, and with reversed order, that is with $F(e_i,e_i) = -q(e_i)$,$F(e_i,e_j) = 0$ for $i < j$ and $F(e_i,e_j) = -\Phi(e_i,e_j)$ for $i > j$. In particular we have $q(x) + F(x,x) = 0$. The map $\bar{\lambda}_F$ of Proposition 1.23 provides now vector space isomorphism $\bar{\lambda}_F : \text{Cl}(0) \to \text{Cl}(q)$. We will now prove that

$$\bar{\lambda}_F(e_H) = \bar{\lambda}_F(e_{i_1}\wedge\cdots\wedge e_{i_n}) = e_{i_1}\cdots e_{i_n}, \tag{92}$$

where the multiplication on the right hand side is that in $\text{Cl}(q)$. The proof of this last property is by induction. It is evident for $n = 1$, since $\bar{\lambda}_F(x) = x$ for every $x \in M$. Suppose it holds for all sequences $i_1 < \cdots < i_n$. We will show that then it also holds for sequences of length $n+1$. We will use the

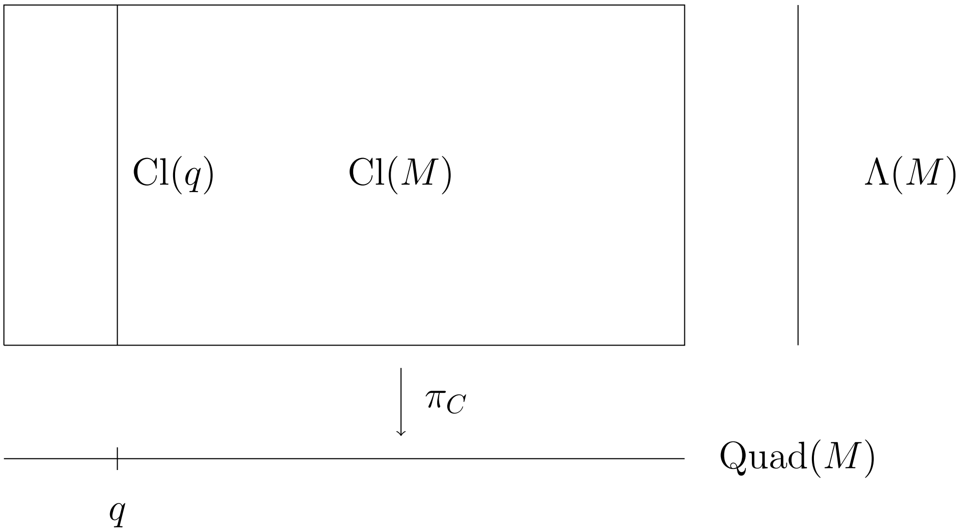

Figure 3: Vector bundle $\mathrm{Cl}(M)$ of Clifford algebras $\mathrm{Cl}(q)$ over the base $\mathrm{Quad}(M)$ of quadratic forms, with the exterior algebra $\Lambda(M)$ as a typical fibre. Global fibre coordinates are provided by selecting a basis in M, as shown in Theorem 3.2

fundamental property of $\bar{\lambda}_F$ in Eq. (33. Suppose H has $n+1$ elements, and let j be its first element, with $H = \{j, i_1, \cdots i_n\}$, and $j < i_1 < \cdots < i_n$. Using Eq. (33 we have

$$\bar{\lambda}_F(e_H) = e_j \bar{\lambda}_F(e_{i_1} \wedge \cdots \wedge e_{i_n}) + i^F_{e_j}(\bar{\lambda}_F(e_{i_1} \wedge \cdots \wedge e_{i_n})). \tag{93}$$

By the induction hypothesis we have $\bar{\lambda}_F(e_{i_1} \wedge \cdots e_{i_n}) = e_{i_1} \cdots e_{i_n}$. Therefore

$$\bar{\lambda}_F(e_H) = e_j e_{i_1} \cdots e_{i_n} + \bar{i}^F_{e_j}(e_{i_1} \cdots e_{i_n}). \tag{94}$$

But now we use Eq. (26) and find the last term vanishes, because expanding it we will be getting terms with $F(e_j, e_{i_k})$ which vanish by the very construction of F. □

3.3 Automorphisms and deformations in the bundle of Clifford algebras

We have arrived at the following picture: We have action, let us denote it by $\tilde{\lambda}$, of the additive group of bilinear forms $\mathrm{Bil}(M)$ on the manifold $\mathrm{Quad}(M)$

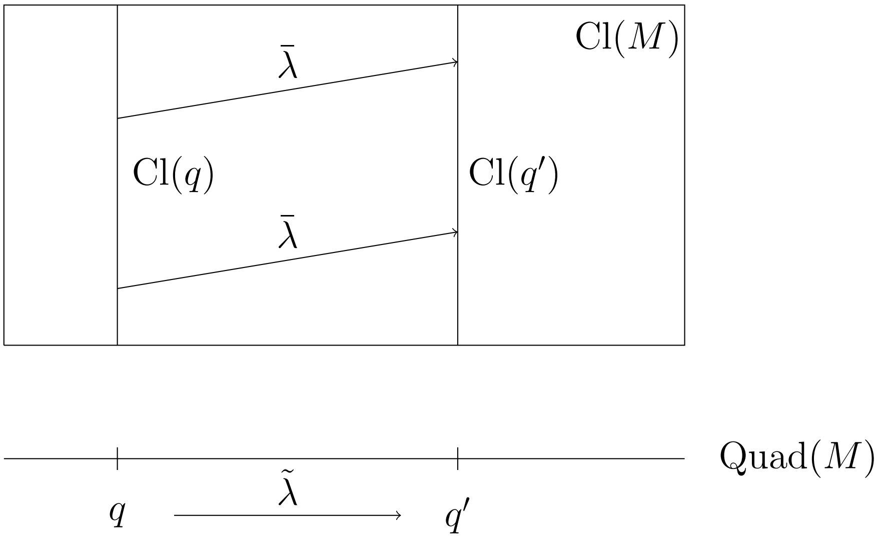

Figure 4: Every bilinear form $F \in \mathrm{Bil}(M)$ defines a automorphism of the vector bundle of Clifford algebras mapping linearly fibers onto fibers. Alternate forms in $\mathrm{Alt}(M) \subset \mathrm{Bil}(M)$ define vertical automorphisms - they do not move points on the base and map every fiber into itself. Such automorphisms are also called *gauge transformations*

the stability subgroup being $\mathrm{Alt}(M)$, the additive group of alternate forms:

$$\begin{array}{rcl} \tilde{\lambda} : \mathrm{Bil}(M) \times \mathrm{Quad}(M) & \rightarrow & \mathrm{Quad}(M) \\ \tilde{\lambda}(F, q) & = & q', \\ q'(x) & = & q(x) - F(x, x). \end{array} \tag{95}$$

[7] The group $\mathrm{Bil}(M)$ acts on the basis on the basis of the vector bundle $\mathrm{Cl}(M)$ whose fibers are Clifford algebras $\mathrm{Cl}(q)$. And we know that his action admits what is called a *lifting*, and we denote it with the letter $\bar{\lambda}$, to the bundle $\mathrm{Cl}(M)$:

$$\bar{\lambda}(F, u) = \bar{\lambda}_{-F}(u),\ u \in \mathrm{Cl}(q), \tag{96}$$

where $\bar{\lambda}_F$ have been defined in Proposition 1.23. Now $\bar{\lambda}(F, C_q) = C(\tilde{\lambda}(q)) = C(q')$. Thus fibers are mapped onto fibers by linear isomorphisms - see Fig. 4. For $F \in \mathrm{Alt}(M)$ we have $q' = q$ and so each fiber $\mathrm{Cl}(q)$ is mapped linearly onto itself.

[7] Here we have defined the action as a subtraction rather than as an addition because of the convention already taken in Proposition 1.23, where $\bar{\lambda}_F$ was defined as a mapping from $C(q')$ to $\mathrm{Cl}(q)$ rather than from $\mathrm{Cl}(q)$ to $C(q')$. Here we are exchanging in our notation q and q'

In each Clifford algebra $\mathrm{Cl}(q)$ we can now define a family of its deformations parameterized by bilinear forms $F \in \mathrm{Bil}(M)$. We do it the same way as we have introduced Clifford algebra structure in the exterior algebra. Given $F \in \mathrm{Bil}(M)$ we define new algebra product $\cdot_F$ in $\mathrm{Cl}(q)$ using the formula:

$$u \cdot_F w = \bar{\lambda}_F(\bar{\lambda}_{-F}(u)\bar{\lambda}_{-F}(w)),\ (u, w \in \mathrm{Cl}(q)). \tag{97}$$

The new product so defined is automatically associative.[8] The formula defining explicitly the new multiplication in $\mathrm{Cl}(q)$ can be derived exactly the same way as we have derived the formula (84:

$$\boxed{x \cdot_F u = xu + \bar{i}^F_x(u).}, \tag{98}$$

where, for $x, y \in M$, $u, w \in \mathrm{Cl}(q)$ we have

$$\boxed{\bar{i}^F_x(yw) = F(x, y)w - y\bar{i}^F_x(w),} \tag{99}$$

and the multiplications on the right in (98) and on the left in (99) are in $\mathrm{Cl}(q)$.

In Ref. [9], in Section 4.7, *Deformations of Clifford algebras*, the formula completely equivalent to Eq. (84) is derived using rather advanced algebraic manipulations and associativity necessitates a complicated almost one page proof.

The formula given in Ref. [9] also involves a certain *exponential*. We will see how exponential enters in our case in a way analogous to our discussion of the mapping λ_F as an exponential.

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[8]It is quite general and almost evident.Let A be a set, B be an algebra, and $T : A \to B$ be a linear map. Let $\cdot$ be the product defined in A as $a \cdot b = T^{-1}(TaTb)$. Then

$$(a \cdot b) \cdot c = T^{-1}(\,T(T^{-1}(TaTb))Tc\,) = T^{-1}(\,TaTbTc\,)$$

, and associativity follows from the associativity of the product in B.

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