Notes on Clifford Algebras

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Contents

Abstract

My notes while studying general Clifford algebras

1 Introduction

1.1 Preliminaries

A reasonably general formulation of the theory of Clifford algebras starts with the definition of the Clifford algebra of a module over a ring, equipped with a quadratic form. A not necessarily symmetric and, in general, degenerate, bilinear form can also appear within this theory. Later on modules are replaced by vector spaces over a field of characteristic zero. We start with the definitions, where we follow the references [6, 15]

Definition 1.1 (Ring)**.** *A ring is a set R with two laws of composition, one denoted additively and the other multiplicatively, which satisfy the following conditions:*

1. *The elements of R form a commutative group under addition;*
2. *The elements of R form a monoid under multiplication;*
3. *If a, b, c are elements of R, we have*

$$a(b + c) = ab + ac,\ (a + b)c = ac + bc.$$

That R is a monoid under multiplication means that

1. $(ab)c = a(bc)$ *for all $a, b, c \in R$ (associativity),*
2. *There is an element $1 \in R$ such that $1a = a1 = a$ for all a in R (that is 1 is the multiplicative identity (neutral element).*

A ring containing at least two elements, in which every nonzero element a has a multiplicative inverse a^{-1} is called a division ring *(sometimes also called a "skew field"). A commutative division ring is called a* field.

Definition 1.2 (Characteristic)**.** *Let R be a ring with unit element* 1*. The* characteristic *of R is the smallest positive number n such that*

$$\underbrace{1 + ... + 1}_{n \text{ summands}} = 0.$$

If such a number does not exist, the characteristic is defined to be 0*.*

We notice that the above condition is equivalent to

$$\underbrace{\alpha + ... + \alpha}_{n \text{ summands}} = 0$$

for every $0 \neq \alpha \in R$.

In applications to Clifford algebras R **will be always assumed to be commutative**. Ultimately R will become the field of real or of complex numbers, but for a while it costs us nothing to be more general. The notation and definitions below follow closely those in Ref. [7].

Definition 1.3 (Module)**.** *Let R be a commutative ring. A* module *over R (in short R-module) is a set M such that*

1. *M has a structure of an additive group,*
2. *For every $\alpha \in R$, $a \in M$ an element $\alpha a \in M$ called scalar multiple is defined, and we have*
   - *i)* $\alpha(x + y) = \alpha x + \alpha y$,
   - *ii)* $(\alpha + \beta)x = \alpha x + \beta x$,
   - *iii)* $\alpha(\beta x) = (\alpha\beta)x$,
   - *iv)* $1 \cdot x = x$.

Remark 1.4. *Normally one would distinguish between left modules and right modules, where multiplication by scalars (element of the ring R) is defined from the left or from the right. But since we will assume that R is commutative, there is no necessity to distinguish between left and right modules. Indeed, in a left module, multiplying x first by α, then by beta, we would get $\beta\alpha x$, thus getting x multiplied by $\beta\alpha$. In a right module doing the same we would get $x\alpha\beta$, thus x multiplied by $\alpha\beta$. In a commutative module $\alpha\beta = \beta\alpha$, therefore it does not matter whether we write the multiplication on the left or on the right.*

1.1.1 Vector spaces

Definition 1.5 (Vector space)**.** *If R is a division ring, then a module M over R is called a* vector space.

In general one would consider left and right vector spaces, but since we assume that R is commutative, it is not necessary to distinguish between the two cases. In linear algebra one shows that every vector space has a basis, possibly infinite, of linearly independent vectors. Moreover, every two bases have the same cardinal number called the dimension of the vector space - cf. e.g. [6, p. 103]. Every system of linearly independent vectors can be extended to a basis.

In particular for every nonzero vector $x \in M$ there exists a linear functional f on the space - that is an element of the dual space M^* - that takes a nonzero value on this vector: $f(x) \neq 0$.

Clifford algebras are usually studied with a restriction to finite dimensional vector spaces. But such a restriction is not necessary at the very beginning, for the study of many important general properties of Clifford algebra.

1.1.2 Algebras

Definition 1.6 (Associative algebra with identity called simply hereafter algebra)**.** *An algebra A over R is a module over R with a multiplication which makes A a ring and satisfying*

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \; (x, y \in A, \, \alpha \in R).$$

Notice that it follows from the definition above, the part where it is said that A is a ring, that the algebra will be always assumed to contain a neutral element, usually denoted as 1.

A subset B of an algebra A is called a *subalgebra* if for any x, y from B, α in R, also $\alpha x, x + y, xy$ are in B, and if B contains the unit 1 of A. A subset S of an algebra A is called a set of generators if A is the smallest subalgebra of A containing S. Notice that a subalgebra must automatically contain the unit of A.

1.1.3 Tensor algebra

Definition 1.7 (Tensor algebra)**.** *Let M be a module over R. An algebra T is called a tensor algebra over M (or "of M") if it satisfies the following* universal property

1. *T is an algebra containing M as a submodule, and it is generated by M,*

2. *Every linear mapping λ of M into an algebra A over R, can be extended to a homomorphism θ of T into A.*

Note 1.8 (Chevalley's construction of the tensor algebra)**.** *In all standard textbooks, see e.g. [4, 7, 15], the above characterisation of the tensor algebra of a module is always completed by a prove of its existence, i.e. by its construction. Chevalley [7] does it in an original way, using the construction of a free algebra as follows.*

Step 1 *First of all given any set $\{x_i\}_{i\in I}$ indexed by an index i in some indexing set I, we can construct an algebra in which this set is the set of linearly and algebraically independent generators. The construction goes as follows. We consider the set Σ of all finite sequences of elements of I. In Σ we include also the empty sequence σ_0 containing no elements from I. With σ_0 we associate the symbol "1". It will become the unit element of our algebra. From theorems of linear algebra we know that there exists a module which has a basis that is equipotent to the set Σ. In other words, there exists a module F in which there is a basis that can be indexed by means of the elements of the set Σ. Given an element $\sigma \in \Sigma$, that is a finite sequence of elements of I we have $\sigma = \{i_1, ..., i_n\}$. Let $y_{\sigma\sigma\in\Sigma}$ be the basis in F. To define the algebra multiplication in F we only need to specify the multiplication of the basis elements. This is defined in a natural way as a juxtaposition $y_\sigma y_{\sigma'} = y_{\sigma\sigma'}$. At the end we can replace every symbol i with the corresponding element of the set $\{x_i\}_{i\in I}$. In this way we obtain* the free algebra with the set $\{x_i\}_{i\in I}$ as the set of generators. *Notice that it follows automatically that the symbol "1" becomes the unit of our algebra, as a juxtaposition of the empty set σ_0 with any σ is σ.*

Step 2 *Let now M be a module. We will construct the tensor algebra $T(M)$ of M. First we consider M as a set, ignoring its module structure. Then we build the free algebra F with M as the set of generators. And now we take into account the existing module structure of M by dividing F by an appropriate two sided ideal as follows. In F we have the algebra structure introduced by its construction. In order to distinguish between the linear operations within F from those within M we denote the addition and subtraction in F by the symbols $\dot{+}$ and $\dot{-}$, and multiplication by scalars by $\alpha \cdot x$. Thus, right now, in F we have, for instance, if x, y are in M, then $x + y \in M$ but, in general, $x \dot{+} y \notin M$, and also $\alpha x \in M$ but $\alpha \cdot x \notin M$. To build the tensor algebra over M we need, for $x, y \in M$, to have $x \dot{+} y = x + y$*

and $\alpha \cdot x = \alpha x$. To this end let S be the set of all elements of the forms:

$$x \dot{+} y \dot{-} (x+y),\ (x, y \in M),$$

and

$$\alpha \cdot x \dot{-} (\alpha x)\, (\alpha \in R, x \in M),$$

and let $\mathcal{T}$ be the two sided ideal in F generated by S. The tensor algebra $T(M)$ of M is then defined as the quotient $F/\mathcal{T}$. Chevalley then shows that $T(M)$ so constructed has the universal property described in Definition 1.7

Let M be a module over R and let $T(M)$ be its tensor algebra. The multiplication within the algebra $T(M)$ inherited from the algebra F is denoted $\otimes$. Since the ideal $\mathcal{T}$ is generated by elements homogeneous of grade 1 in M, the resulting algebra $T(M)$ is also graded. We have

$$T(M) = \bigoplus_{p=0}^{\infty} T^p M, \tag{1}$$

where

$$T^p M = M^{\otimes p} = \underbrace{M \otimes ... \otimes M}_{p \text{ factors}}. \tag{2}$$

It is understood here that $T^0 M = R$ and $T^1 M = M$. The tensor algebra is a graded and associative (but non-commutative) algebra, with unit $1 \in R$. The fact that $T(M)$ is a graded algebra means that for any $x \in T^p M, y \in T^q M$ the product xy is in $T^{p+q} M$ for all $p, q = 0, 1, \cdots$. Sometimes it is convenient to consider $T^p M$ for $p < 0$ as consisting of the zero vector only.

1.1.4 Quadratic forms

Given a module M over a ring R we will define now quadratic form on M. There are two definitions possible, one more general than the other one if general rings with any characteristic are being considered. Bourbaki [4] and Chevalley [7] use the more general definition adapted to a general case. Below I will give an example of how careful one has to be in a general case, I will closely follow the monograph by Helmstetter [10].

Definition 1.9 (Quadratic form I)**.** *Let M be a module over a commutative ring R. A mapping $q : M \to R$ is called a quadratic form on M if the following conditions are satisfied:*

1.

$$q(\alpha x) = \alpha^2\, q(x)\, \textit{for all } \alpha \in R,\, x \in M, \tag{3}$$

2. *There exists a bilinear form $\Phi(x,y)$ on M such that for all $x,y \in M$ we have*

$$\Phi(x,y) = q(x+y) - q(x) - q(y). \tag{4}$$

We say that the bilinear form Φ is associated with the quadratic form q. Sometimes Φ is also called the polar form of q. *It follows from its very definition that Φ is* symmetric*: $\Phi(x,y) = \Phi(y,x)$ for all $x,y \in M$.*

We can combine Eqs. (4) and (3) into:

$$q(\alpha x + \beta y) = \alpha^2 q(x) + \beta^2 q(y) + \alpha\beta\Phi(x,y). \tag{5}$$

The short discussion of consequences given below is taken directly from Ref. [10].

Note 1.10. *From the very definition we find that*

$$\Phi(x,x) = q(2x) - 2q(x) = 4q(x) - 2q(x) = 2q(x). \tag{6}$$

It follows that if R is of characteristic 2, then $\Phi(x,x) = 0$ for all $x \in R$. Such a form is called alternate. *In that case, since also $\Phi(x+y,x+y) = 0$, we have that*

$$\begin{aligned} 0 = \Phi(x+y,x+y) &= \Phi(x,x) + \Phi(x,y) + \Phi(y,x) + \Phi(y,y) \\ &= \Phi(x,y) + \Phi(y,x), \end{aligned} \tag{7}$$

so that in this case the form Φ is antisymmetric $\Phi(x,y) = -\Phi(y,x)$.

Getting back to a general characteristic, we may also notice at this point that *if the mapping $x \mapsto 2x$ is surjective* in M, then the form Φ determines q. Indeed, setting $y = 2x$ we get $q(y) = q(2x) = 4q(x) = 2\Phi(x,x)$. We also observe that the quadratic form q is determined by the associated bilinear form Φ when the mapping $\alpha \mapsto 2\alpha$ is injective in R, in other words if multiplication by $\frac{1}{2}$ makes sense in R. In that case we can solve Eq. (6) to obtain $q(x) = \frac{1}{2}\Phi(x,x)$.

In applications to Clifford algebras, unless we are interested in very special cases like characteristic 2, it is more convenient to use a little bit different definition of a quadratic form, as given, for instance, in Ref. [16, p. 199]:

Definition 1.11 (Quadratic form II)**.** *Let M be a module over a commutative ring R. A function $q : M \to R$ is called a quadratic form if there exists a bilinear form $F : M \times M \to R$ such that*

$$q(x) = F(x,x). \tag{8}$$

It follows from this last definition that the condition in Eq.(3) is then automatically satisfied, and also the condition in Eq.(4) is automatically satisfied with

$$\Phi(x,y) = F(x,y) + F(y,x). \tag{9}$$

Remark 1.12. *If the module M admits a basis (in particular, when it is a vector space), then given a quadratic form q as in Def. 1.9 one can always construct a bilinear form F (in general a non symmetric one) such that $q(x) = F(x,x)$ (cf. eg. Ref. [5, Proposition 2, p. 55]). It is instructive to understand the idea of the proof (taken from Ref. [5, Proposition 2, p. 55]).* [1] *Of course if the field R admits division by 2, we can use Eq. (6) and simply set $F(x,y) = \frac{1}{2}\Phi(x,y)$. In particular the rest of this remark is irrelevant for vectors spaces over reals or complex number fields*

Let q be a quadratic form on a vector space M, and let Φ be the associated bilinear form. We start with noticing that M, being a vector space, has a basis $\{e_i\}_{i\in I}$. By the well-ordering theorem every set can be well ordered, and we will assume that the index set I is well ordered. Since $\{e_i\}_{i\in I}$ is a basis, every bilinear form F is uniquely determined by the coefficients f_{ij}, $i,j \in I$. Let Φ be the bilinear form associated to q. We first observe that if $\{\alpha_i\}_{i\in I}$ is any family of elements of R with only a finite number of $\alpha_i \neq 0$, then

$$q(\sum_i \alpha_i e_i) = \sum_i \alpha_i^2 q(e_i) + \sum_{\{i,j\}} \alpha_i \alpha_j \Phi(e_i, e_j), \tag{10}$$

where the last sum is over all two-element subsets of I. [2]

It is understood that each sum is over a finite set determined by non-zero α_i-s. We prove Eq. (10) by induction with respect to the number n of nonzero coefficients α_i. If there are only two nonzero coefficients, then (10) follows from Eq. (5), i.e. from the definition of the quadratic form 1.9. Assume now that Eq. (10) holds for subsets $\{i_1, ..., i_n\}$ of n non-zero coefficients α_i, and let us add another non-zero coefficient $\alpha_{i_{n+1}}$. Then

$$q(\alpha_{i_1}e_{i_1} + ... + \alpha_{i_{n+1}}e_{i_{n+1}}) = q\left((\alpha_{i_1}e_{i_1} + ... + \alpha_{i_n}e_{i_n}) + \alpha_{i_{n+1}}e_{i_{n+1}}\right)$$
$$= q(\alpha_{i_1}e_{i_1} + ... + \alpha_{i_n}e_{i_n}) + q(\alpha_{i_{n+1}}e_{i_{n+1}}) + \Phi(\alpha_{i_1}e_{i_1} + ... + \alpha_{i_n}e_{i_n}, \alpha_{i_{n+1}}e_{i_{n+1}}).$$

Using now the quadratic form property, in particular for the sum of two elements, the assumed property for the sum of n elements, as well as linearity

[1] The proof can be also found in Ref. [I.2.2, p. 76][7], but with unnecessary assumption that M is finite dimensional.

[2] Thus if a_{i_1} and a_{i_2} are nonzero, with $i_1 < i_2$, then only $\Phi(e_{i_1}, e_{i_2})$ enters the sum, and not $\Phi(e_{i_2}, e_{i_1})$ because $\{i_2, i_1\}$ is the same subset as $\{i_1, i_2\}$

of Φ in the first argument leads to the desired result. Nowhere we needed to assume that the basis $\{e_i\}_{i\in I}$ is finite.

Given a quadratic form q we can now define a bilinear form F satisfying $q(x) = F(x,x)$ by defining its coefficients f_{ij}, $i,j \in I$, as follows:

$$\begin{aligned} f_{ii} &= q(e_i), && (11)\\ f_{ij} &= \Phi(e_i,e_j),\ i<j, && (12)\\ f_{ij} &= 0,\ i>j. && (13)\end{aligned}$$

We now check that $q(x) = F(x,x)$ for every x in M. If $x \in M$ then $x = \sum_i \alpha_i e_i$, with only a finite number of non-zero terms in the sum. Therefore, using Eq. (10) we have

$$q(x) = \sum_i \alpha_i^2 q(e_i) + \sum_{i<j} \alpha_i\alpha_j \Phi(e_i,e_j). \tag{14}$$

On the other hand, from bilinearity of F we have

$$F(x,x) = F(\sum_i \alpha_i e_i, \sum_j \alpha_j e_j) = \sum_i \alpha_i^2 f_{ii} + \sum_{i\neq j} \alpha_i\alpha_j f_{ij} = q(x)$$

from the definition of the coefficients f_{ij} above (because $f_{ij} = 0$ for $i > j$) .

1.1.5 Diagonalization of symmetric bilinear forms

When studying Clifford algebras it is often convenient to use particularly nice properties of orthogonal bases for symmetric bilinear forms. Such a basis always exists for finite dimensional vector spaces over the field of characteristic different from 2, and it is instructive to look at the proof of the proposition below (taken from Ref. ([9, p. 362]), cf. also ([12]).

Proposition 1.13. *Let M be a finite dimensional vector space over a field of characteristic $\neq 2$ and let F be a symmetric bilinear form on M. Then there exists a basis $\{e_i\}$, $(i = 1, ..., n)$ in M consisting of mutually orthogonal vectors: $F(e_i,e_j) = 0$ for $i \neq j$. In other words F is diagonalizable.*

Proof. The proof is by induction with respect to the dimension n of the vector space. The statement is trivially true for $n = 1$, since in this case the set $i \neq j$ is empty. Suppose the statement holds for vector spaces of dimension $n-1$ or less. We will show that then it holds also for dimension n. For this we will need a little auxiliary results, and it is in the proof of this auxiliary result we will use the fact that the characteristic is $\neq 2$. Namely we first need to show that if the symmetric bilinear form is nontrivial, i.e.

$F \neq 0$, then there always exists a vector x such that $F(x,x) \neq 0$. The fact that $F \neq 0$ is equivalent to saying that there exist vectors u, v for which $F(u,v) \neq 0$. If $F(u,u) \neq 0$ or $F(v,v) \neq 0$, we are done, but if $F(u,u) = 0$ and $F(v,v) = 0$, then $x = u + v$ does the job. Indeed then $F(x,x) = F(u,u) + F(v,v) + F(u,v) + F(v,u) = 2F(u,v)$ since we have assumed that F is symmetric. But, since we also assume that the filed is not of characteristic 2, then $2 \neq 0$, and therefore $F(x,x) \neq 0$.

Let us return now to the proof of the main statement, assuming M n-dimensional. If $F = 0$ any basis does the job. Let us therefore assume that $F \neq 0$. Then, as we have just shown, there exists a vector x such that $F(x,x) \neq 0$. Evidently $x \neq 0$. Then we define W as the following subspace of M (the orthogonal complement of x).

$$W = \{w \in M : F(x,w) = 0\}. \tag{15}$$

Evidently W is a vector space that does not contain x. Moreover we have that every vector $v \in M$ can be uniquely written in the form $v = w + \alpha x$, where $w \in W$ and α is a scalar. For if v is in W we set $\alpha = 0$ and if $v \notin W$, then $F(x,v) \neq 0$, and it is enough to set $\alpha = F(x,v)/F(x,x)$ and define $w = v - \alpha x$. Then automatically $F(w,x) = 0$ i.e. $w \in W$, and $v = w + \alpha x$.

If $\{e_i\}$ is a basis in W, then $\{e_i\} \cup \{x\}$ is a basis in M. Therefore W is $n-1$-dimensional and, by the induction hypothesis, there exists a basis $e_1, ..., e_{n-1}$ in W diagonalizing F. But then e_i together with x is a basis in M, and it is diagonalizing F, since $e_i \in W$ and therefore, by the definition of W, $F(x, e_i) = 0$ for $i = 1, ..., n$. □

1.1.5.1 Degenerate and nondegenerate bilinear forms

With the assumptions and notation as above, let F be a bilinear form on M, *but not necessarily symmetric*. When F is not necessarily symmetric, there are two possible definitions of a *degenerate bilinear form*:

(i) There exists $y \in M$, $y \neq 0$ such that $F(x,y) = 0 \, \forall x \in M$;

(ii) There exists $x \in M$, $x \neq 0$ such that $F(x,y) = 0 \, \forall y \in M$;

But in fact the two conditions are equivalent, and each of them is equivalent to the conditions that the matrix $\mathbf{F}_{ij} = F(e_i, e_j)$ is not invertible.

Indeed (i) is equivalent to: there exists $y \in M$, $y \neq 0$, such that $F(e_i, y) = 0$ for all $i = 1, ..., n$. Let us write $y = \sum_{j=1}^{n} y^j e_j$. Then $F(e_i, y) = 0$ can be written as $\sum_{j=1}^{n} F(e_i, e_j) y^j = 0$, or, in matrix notation, $\mathbf{F}\mathbf{y} = 0$, which is another way of saying that the matrix $\mathbf{F}$ is not invertible. The condition (ii) would lead to the same conclusion, but for the transposed matrix. But the matrix is invertible if and only if the transposed is invertible (inverse of the transpose is the transpose of the inverse [4, p. 350]), which shows the equivalence of (i) and (ii). The bilinear form that is not degenerate is called *nondegenerate.*

Remark 1.14. *We notice that a symmetric bilinear form F and an* orthogonal basis *e_i, F is non-degenerate if and only if all the diagonal elements $F(e_i, e_i)$ are non-zero. In fact, if F is nondegenerate, then all $F(e_i, e_i)$ must be non zero, since if one of them vanishes, $F(e_i, e_i) = 0$, then this e_i is orthogonal to all vectors in M. Conversely, if F is degenerate and there exists a non-zero x such that $F(x, e_i) = 0$ for all i, then one of the terms $F(e_i, e_i)$ must be zero. Indeed writing $x = \sum_j x_j e_j$ we find that $0 = F(x, e_i) = \sum_j x_j F(e_j, e_i) = x_i F(e_i, e_i)$, because of the orthogonality of the basis. If one of the coefficients x_i is non-zero, then $F(e_i, e_i) = 0$.*

1.2 Clifford algebras - definition

Let q be a quadratic form on M (see Def. 1.9), and let $J(q)$ be the two-sided ideal in $T(M)$ generated by elements of the form $x \otimes x - q(x)1$, where $x \in M \subset T(M)$. The ideal $J(q)$ consists of all finite sums of elements of the form $x_1 \otimes \ldots \otimes x_p \otimes (x \otimes x - q(x)1) \otimes y_1 \otimes \ldots \otimes y_q$, where $x, x_1, \ldots, x_p, y_1, \ldots, y_q$ are in M.

Definition 1.15 (Clifford algebra, cf. [7, p. 35])**.** *With M and q as above the quotient algebra $\mathrm{Cl}(q) = T(M)/J(q)$ is called the Clifford algebra associated to M and q.*

Denoting by $\pi_q : T(M) \to \mathrm{Cl}(q)$ the canonical mapping, $\pi_q(M)$ is a submodule of $\mathrm{Cl}(q)$ that generates $\mathrm{Cl}(q)$ as an algebra. Moreover, for all $x \in M$ we have

$$(\pi_q(x))^2 = q(x)1. \tag{16}$$

From $\pi_q(x + y)^2 - \pi_q(x)^2 - \pi_q(y)^2 = q(x + y) - q(x) - q(y) = \Phi(x, y)$ we find that

$$\pi_q(x)\pi_q(y) + \pi_q(y)\pi_q(x) = \Phi(x, y). \tag{17}$$

If M is a vector space, then the mapping $x \mapsto \pi_q(x)$ is injective (which will be shown later) and M can be identified with a linear subspace of $\mathrm{Cl}(q)$. In general it needs not be so. The case of $q = 0$ is special. The Ideal $J(q)$ is then generated by homogeneous elements $x \otimes x$ and the algebra $\mathrm{Cl}(0)$ is nothing but *the exterior algebra* $\Lambda(M)$ of M. All homogeneous elements of $J(0)$ are then of at least the degree 2, therefore no non-zero element of M can belong to this ideal. It follows that in this case the mapping $x \mapsto \pi(x)$

is an embedding and M can be always identified with the grade 1 subspace of Cl(0).

1.2.1 Universal property

The Clifford algebra $\mathrm{Cl}(q)$ defined above is characterized by a universal property analogous to the universal property characterizing the tensor algebra as defined in Definition 1.7.

Theorem 1.16 (Cf. e.g. [7, Theorem 3.1, p. 36])**.** *Assume that λ is a linear mapping from M into an algebra A with the property that $(\lambda(x))^2 = q(x)1$ for all x in M. Then there is a unique homomorphism ϕ of algebras over R, with units, such that for all x in M we have*

$$\lambda = \phi \circ \pi. \tag{18}$$

■

1.2.2 Main involution α and main anti-involution τ

It is by using this universal property that one defines the main involution α and the main anti-involution τ of $\mathrm{Cl}(q)$. To define α let λ be the map $\lambda : M \to \mathrm{Cl}(q)$ defined by $\lambda(x) = \pi(-x)$. Evidently

$$(\lambda(x))^2 = (\pi(-x))^2 = (-\pi(x))^2 = q(x).$$

Therefore λ defines ("extends to") a unique algebra homomorphism $\alpha : \mathrm{Cl}(q) \to \mathrm{Cl}(q)$ such that $\alpha(\pi(x)) = \pi(-x) = -\pi(x)$. It follows that

$$\alpha(\alpha(\pi(x))) = \pi(x)$$

thus $\alpha^2 \circ \pi = Id$ on M. From the uniqueness of the extension it follows then that $\alpha^2 = Id$, so that α is an involutive automorphism of $\mathrm{Cl}(q)$. It is called *the main involution*, or *the main automorphism.* Using the universal property in a similar but a somewhat different way one introduces *the main anti-involution* τ. Let $\mathrm{Cl}(q)^{op}$ denote the algebra opposite to $\mathrm{Cl}(q)$. That is $\mathrm{Cl}(q)^{op}$ is the same as $\mathrm{Cl}(q)$ as a linear space, but the multiplication is defined in the opposite order. The product xy in $\mathrm{Cl}(q)^{op}$ is the same as yx in $\mathrm{Cl}(q)$. But squares x^2 are evidently the same in both algebras. The identity map $\iota : x \mapsto x$ from $\mathrm{Cl}(q)$ to $\mathrm{Cl}(q)^{op}$ is an anti-homomorphism, $\iota(xy) = yx$. Consider the map $\lambda : M \to \mathrm{Cl}(q)^{op}$ defined as $\lambda(x) = \iota(\pi(x))$. Since the squares are the same in both algebras, we have that $(\lambda(x))^2 = q(x)1$. Therefore λ extends to an algebra homomorphism from $\mathrm{Cl}(q)$ to $\mathrm{Cl}(q)^{op}$.

Composing this map with the inverse of ι we get $\tau : \mathrm{Cl}(q) \to \mathrm{Cl}(q)$. Arguing as in the previous case we deduce that $\tau^2 = Id$, therefore τ is an anti-automorphism of $\mathrm{Cl}(q)$. From the very definition we have that $\tau(\pi(x)) = \pi(x)$ for all $x \in M$. Since $\pi(M)$ generates $\mathrm{Cl}(q)$, this last property determines the anti-automorphism τ of $\mathrm{Cl}(q)$ uniquely.

Remark 1.17. *For a in $\mathrm{Cl}(q)$ we often write a^τ instead of $\tau(a)$.*

While the tensor algebra $T(M)$ is Z-graded, where Z stands for the Abelian group (under addition) of integers, the quotient algebra $\mathrm{Cl}(q) = T(M)/J(q)$ is only Z_2-graded. That is because the expressions $x \otimes x - q(x)1$ generating the ideal $J(q)$ are not grade homogeneous (unless $q = 0$, in which case $\mathrm{Cl}(q)$ is the exterior algebra of M).

1.2.2.1 The even subalgebra

There is another way of getting to the main automorphism α. Every element of the tensor algebra is a sum of even and odd tensors (that is tensors of even and odd degrees)

$$T(M) = T(M)_{\mathrm{even}} \oplus T(M)_{\mathrm{odd}}. \tag{19}$$

In the tensor algebra $T(M)$ the mapping $x \mapsto -x$ generates algebra automorphism, let us call it $\tilde{\alpha}$, that changes the sign of elements of $T(M)_{\mathrm{odd}}$. Since the expressions $x \otimes x - q(x)$ generating the ideal $J(q)$ are invariant under the transformations $x \mapsto -x$, the automorphism $\tilde{\alpha}$ of the tensor algebra descends to the quotient algebra $\mathrm{Cl}(q)$. It maps $\pi_q(x)$ into $-\pi_q(x)$, therefore it coincides with the main automorphism α. It follows that α simply changes the sign of products of odd numbers of $\pi_q(x)\, x \in M$.

Note 1.18. *There is no standard notation for the main automorphism and anti-automorphism. Different authors use different letters to denote them.*

We now define $\mathrm{Cl}(q)^+ = \pi_q(T(M)_{\mathrm{even}}), \mathrm{Cl}(q)^- = \pi_q(T(M)_{\mathrm{odd}})$, and we obtain the direct sum decomposition

$$\mathrm{Cl}(q) = \mathrm{Cl}(q)^+ \oplus \mathrm{Cl}(q)^-, \tag{20}$$

where $\mathrm{Cl}(q)^+$ (resp. $\mathrm{Cl}(q)^-$) is generated by sums of even (resp. odd) number of elements of $\pi_q(M)$.

Notice that the product of any two even elements is even, the product of any two elements one of which is odd and one even, is odd, and the product of any two odd elements is even. I short:

$$\begin{aligned} \mathrm{Cl}(q)^+\mathrm{Cl}(q)^+ \subset \mathrm{Cl}(q)^+,\ \mathrm{Cl}(q)^+\mathrm{Cl}(q)^- \subset \mathrm{Cl}(q)^-, \\ \mathrm{Cl}(q)^-\mathrm{Cl}(q)^- \subset \mathrm{Cl}(q)^+. \end{aligned} \tag{21}$$

Therefore (since also 1 is an even element) $\mathrm{Cl}(q)^+$ is a subalgebra of $\mathrm{Cl}(q)$. It is called *the even Clifford algebra.*

1.2.3 Anti-derivations

We denote by M^* the dual module, that is the module of all linear functions from M to R.

Lemma 1.19 ([7, Lemma 3.2, p. 43],[5, Lemma 1, p. 141])**.** *Let f be an element of M^*. There exists a unique linear mapping i_f from $T(M)$ to $T(M)$ such that*

1. *We have*
$$i_f(1) = 0, \tag{22}$$
2. *For all $x \in M \subset T(M)$, $u \in T(M)$. we have*
$$i_f(x \otimes u) = f(x)u - x \otimes i_f(u). \tag{23}$$

The map $f \mapsto i_f$ from M^* to linear transformations on $T(M)$ is linear. We have

(i) $i_f(T^pM) \subset T^{p-1}M$,

(ii) $i_f^2 = 0$,

(iii) $i_f i_g + i_g i_f = 0$, *for all $f, g \in M^*$.*

If q is a quadratic form on M then the ideal $J(q)$ is stable under i_f, that is $i_f(J(q)) \subset J(q)$, and thus i_f defines the mapping, denoted by $\bar{i}_f$, on the quotient Clifford algebra $\mathrm{Cl}(q) = T(M)/J(q)$*:*
$$\pi_q \circ i_f = \bar{i}_f \circ \pi_q. \tag{24}$$
On $\mathrm{Cl}(q)$ *we then have*

(iv) $\bar{i}_f(1) = 0$, $(1 \in \mathrm{Cl}(q))$

(v) *For all $x \in M$, $w \in \mathrm{Cl}(q)$, we have*
$$\bar{i}_f(\pi_q(x)w) = f(x)w - \pi_q(x)\bar{i}_f(w). \tag{25}$$

■

Corollary 1.20. *[7, Corollary, p. 44] If M is a vector space, then the mapping $\pi_q : M \to \mathrm{Cl}(q)$ is injective and we can identify M with $\pi_q(M)$.*

The proof goes as follows. Let x be a nonzero vector in M and let f be an element from M^* for which $f(x) = 1$ (cf. Section 1.1.1). Let i_f be as in Lemma 1.19. Setting $w = 1$ in Eq. 25 we get $i_f(\pi_q(x)) = f(x)1 \neq 0$, therefore $\pi_q(x) \neq 0$.

1.2.4 Bourbaki's application λ_F

Definition 1.21. *Let F be a bilinear form on M. Then every $x \in M$ determines a linear form f_x on M defined as $f_x(y) = F(x, y)$. We will denote by $\bar{i}_x^F$ the antiderivation $\bar{i}_{f_x}$ described in Lemma 1.19. In particular we have:*

(i) $\bar{i}_x^F(1) = 0$, $(1 \in \mathrm{Cl}(q))$

(ii) *For all $y \in M$, $w \in \mathrm{Cl}(q)$, we have*

$$\bar{i}_x^F(yw) = F(x, y)w - y\,\bar{i}_x^F(w). \tag{26}$$

Proposition 1.22. *With the notation as in the Definition 1.21, for $y_1, ..., y_n$ in $\mathrm{Cl}(q)$ we have*

$$\bar{i}_x^F(y_1...y_n) = \sum_{j=1}^{n}(-1)^{n-1}F(x, y_j)y_1...\hat{y}_j...y_n, \tag{27}$$

where $\hat{y}_j$ means that this entry is omitted in the product.

In particular if $F(x, y_j) = 0$ for all $j = 1, .., n$, then $\bar{i}_x^F(y_1...y_n) = 0$.

Proof. The proof follows immediately from the definition by induction. □

The following Lemma is taken from Bourbaki [5, p. 142-143]. As we will see it has far reaching consequences.

Lemma 1.23. *There exists a unique linear mapping $\lambda_F : T(M) \to T(M)$ such that*

$$\lambda_F(1) = 1, \tag{28}$$

$$\lambda_F(x \otimes u) = i_x^F(\lambda_F(u)) + x \otimes \lambda_F(u),\ x \in M. \tag{29}$$

For all $f \in M^*$ we have

$$\lambda_F \circ i_f = i_f \circ \lambda_F. \tag{30}$$

If F and G are two bilinear forms on M, then

$$\lambda_F \circ \lambda_G = \lambda_{F+G}. \tag{31}$$

For every bilinear form F the linear mapping $\lambda_F : T(M) \to T(M)$ is a bijection. ■

The consequence of this Lemma for Clifford algebras is described in the following Proposition.

Proposition 1.24 ([5, Proposition 3, p. 13]). *Let q and q' be two quadratic forms on M such that $q'(x) = q(x)+F(x,x)$, where $F(x,y)$ is a bilinear form. The mapping λ_F maps the ideal $J(q)$ onto the ideal $J(q)$ and it defines an isomorphism, denoted $\bar{\lambda}_F$ of the R-module $\mathrm{Cl}(q')$ onto the R-module $\mathrm{Cl}(q)$:*

$$\pi_q \circ \lambda_F = \bar{\lambda}_F \circ \pi_{q'}. \tag{32}$$

Note 1.25. In the following we will always assume that M is a vector space. *Therefore, in particular, M can be identified with $\pi_q(M) \subset \mathrm{Cl}(q)$.*

Proposition 1.26. *For all $x \in M$, $w \in \mathrm{Cl}(q)$ we have*

$$\begin{aligned} \bar{\lambda}_F(1) &= 1, \\ \bar{\lambda}_F(x) &= x, \\ \bar{\lambda}_F(xw) &= \bar{i}^F_x(\bar{\lambda}_F(w)) + x\bar{\lambda}_F(w). \end{aligned} \tag{33}$$

If F, G are bilinear forms, if $q''(x) = q'(x) + G(x,x)$ and $q'(x) = q(x) + F(x,x)$, then

$$\bar{\lambda}_{F+G} = \bar{\lambda}_F \circ \bar{\lambda}_c G. \tag{34}$$

Since $\bar{\lambda}_0$ is the identity map, we thus have

$$(\bar{\lambda}_F)^{-1} = \lambda_{(-F)}. \tag{35}$$

Proof. The only property in the two propositions above that is not taken directly from Ref. [5] is the formula (33). But it follows immediately by applying π_q to both sides of Eq. (29) and making use of Eqs. (25) and (32). Eq. (34) follows directly from Eq. (31) and the definition of the quotient mappings, as we have the general property that *composition of two quotient mappings is the quotient of their composition.* ∎

Note 1.27. *Notice that in Eq. (33) the multiplication xw on the left is in the algebra $C(q')$, while the multiplication $x\bar{\lambda}_F(w)$ on the right is in the algebra $\mathrm{Cl}(q)$.*

Lemma 1.28. *Let M be a vector space. If $x_1, ..., x_n$ are in M and if $F(x_i, x_j) = 0$ for $i < j$, then*

$$\bar{\lambda}_F(x_1...x_n) = x_1...x_n. \tag{36}$$

Proof. The proof is by induction. For $n = 1$ the statement evidently holds. Let us assume it holds for n and suppose we add x such that $F(x, x_i) = 0$ for $i = 1, ..., n$. Then, using Eq. (33) we have

$$\bar{\lambda}_F(xx_1...x_n) = \bar{i}^F_x(x_1...x_n) + xx_1...x_n.$$

But then, using Proposition 1.22, we get $\bar{i}^F_x(x_1...x_n) = 0$. $\square$

The following immediate corollary can be found in Bourbaki [3, Exercise 3c, p. 154]

Corollary 1.29. *Let M be a vector space over a field of characteristic $\neq 2$, q a quadratic form, and Φ the associated bilinear form. Let $F(x, y) = \frac{1}{2}\Phi(x, y)$, so that $q(x) = F(x, x)$, and denote $\mu_q = \bar{\lambda}_F$, so that $\mu_q : \mathrm{Cl}(q) \to \Lambda(M)$. If $x_1, ..., x_n$ are in M and if they are pairwise orthogonal, i.e. $F(x_i, x_j) = 0$ for $i \neq j$ then*

$$\mu_q(x_1...x_n) = x_1 \wedge .... \wedge x_n. \tag{37}$$

1.2.4.1 The mapping λ_F as an exponential Here we assume that the ring R is the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. Replacing F by tF and G by sF, $t, s \in \mathbb{R}$ from the property (31) we obtain

$$\lambda_{tF} \circ \lambda_{sF} = \lambda_{(t+s)F}. \tag{38}$$

Since $\lambda_0 = Id$, it follows that there exists a linear operator a_F on $T(M)$ such $\lambda_{tB} = \exp(ta_F)$. We can find the properties defining a_F by replacing F by tF in Eqs. (28) and (29) defining λ_F, and differentiating with respect to t at $t = 0$. We notice that $\lambda_{tF}(x) = x$ and that, since λ_F is linear in F, we have $i^{tF}_x = ti^F_x$. Taking all this into account we obtain:

$$\begin{aligned} a_F(1) &= 0, && (39)\\ a_F(x) &= 0,\ x \in M, && (40)\\ a_F(x \otimes u) &= i^F_x(u) + x \otimes a_F(u). && (41) \end{aligned}$$

In particular we get

$$\begin{aligned} a_F\,(x \otimes y) &= F(x, y) && (42)\\ a_F\,(x \otimes y \otimes z) &= F(x, y)z - F(x, z)y + F(y, z)x, && (43) \end{aligned}$$

$$\begin{aligned} a_F\,(x \otimes y \otimes z \otimes u) = F(x, y)z \otimes u - F(x, z)y \otimes u + F(y, z)x \otimes u +\\ F(z, u)x \otimes y - F(y, u)x \otimes z + F(x, u)y \otimes z. \end{aligned} \tag{44}$$

Note 1.30. *There is an important particular case when the bilinear form* F *is antisymmetric:* $F(x,y) = -F(y,x)$*. In this case* $q'(x) = q(x) + F(x,x) = q(x)$*. Therefore* $\bar{\lambda}_F$ *maps every* $\mathrm{Cl}(q)$ *into itself. In particular it maps into itself the exterior algebra* $\Lambda(M)$ *of* M*. Thus we can rewrite the equations (42)-(44) replacing* $\otimes$ *by* $\wedge$*.*

In quantum physics exterior algebra is used to describe the Fock space of a Fermi field. The operator a_F *removes two particles from a multiparticle state - it acts like a annihilation of a pair operator. Pairs of Fermions seem to be of some importance in theories of superconductivity. Thus it may be speculated that operators similar to* λ_F *and* a_F *may be relevant for mathematical models of physical phenomena similar to superconductivity.*

1.3 Graded structure of a Clifford algebra

Here we assume that M **is a finite dimensional vector space over a field with characteristic** $\neq 2$.

Remark 1.31. *If* $e_1, ..., e_n$ *is a basis in* M*, then the tensor algebra* $T(M)$ *has the basis* $1, e_i, e_{i_1} \otimes e_{i_2}, ..., e_{i_1} \otimes ... \otimes e_{i_p}, ...$ *Thus a general element of the tensor algebra can be represented as a finite sequence of tables* $t, t^i, t^{i_1 i_2}, ..., t^{i_1 \ldots i_p}$ *where* $t, t^i, t^{i_1 \ldots i_p}$ *(with* $i, i_1, ..., i_p = 1, ..., n$*) are scalars. In the Clifford algebra we skip the symbol of tensor multiplication and we restrict ourselves to* $i_1 < ... < i_p$*, with* $p \leq n$*. The tensor algebra is always infinite dimensional, the Clifford algebra is always of the dimension* 2^n*.*

We will be using the notation as in Corollary 1.29. In particular $\Lambda(M)$ is the exterior algebra over M, q is a quadratic form, F is the unique symmetric bilinear form such $q(x) = F(x,x)$, and $\mu_q = \bar{\lambda}_F$ is the vector space isomorphism $\mu_q : \mathrm{Cl}(q) \to \Lambda(M)$ with the properties that $\mu_q(1) = 1$, $\mu_q(x) = x$ for $x \in M$, and

$$\mu_q(xw) = i_x^F(\mu_q(w)) + x \wedge \mu_q(w). \tag{45}$$

In particular if $x_1, ..., x_n$ are pairwise orthogonal, i.e $F(x_i, x_j) = 0$ for $i \neq j$, then

$$\mu_q(x_1 ... x_n) = x_1 \wedge ... \wedge x_n. \tag{46}$$

From Proposition 1.13 we know that M admits an orthogonal basis $\{e_i\}$,$i = 1, ..., n$. We choose this basis, then μ_q maps each product $e_{i_1} ... e_{i_p}$ in $\mathrm{Cl}(q)$ to the product $e_{i_1} \wedge ... \wedge e_{i_p}$ in the exterior algebra $\Lambda(M)$. The exterior algebra $\Lambda(M)$ is Z-graded:

$$\Lambda(M) = \bigoplus_{p=0}^{n} \Lambda^p(M), \tag{47}$$

where $\Lambda^p(M)$ is $\binom{n}{p} = \frac{n!}{p!(n-p)!}$ dimensional, and the elements $e_{i_1} \wedge ... \wedge e_{i_p}$ (p-vectors) with $i_1 < ... < i_p$ form a basis in $\Lambda^p(M)$. For $p > n$ we have $\Lambda^p(M) = \{0\}$, for $p = 1$ we have $\Lambda^1(M) = M$, and for $p = n$ we have that $\Lambda^n(M)$ is 1-dimensional, spanned by $e_1 \wedge ... \wedge e_n$. For $p = 0$ we have that $\Lambda^0(M)$ is the basic field. The whole exterior algebra $\Lambda(M)$ is $\sum_{p=0}^{n} \binom{n}{p} = 2^n$ dimensional.

We can use the linear isomorphism μ_q to transfer the graded structure of the exterior algebra back to $\mathrm{Cl}(q)$ by defining

$$C^p(q) = \mu_q^{-1}(\Lambda^p(M)). \tag{48}$$

The subspaces $C^p(q)$ are $\binom{n}{p}$-dimensional. Moreover,if $e_1, ..., e_n$ is *any* orthogonal basis for M, then the products $e_{i_1}...e_{i_p}$, $(i_1 < i_2 < ... < i_p)$, form a basis $C^p(q)$.

Remark 1.32. *One has to be careful here. While it is true that any set of linearly independent vectors can be extended to a basis, it is not true, in general, that any set of mutually orthogonal vectors can be extended to an orthogonal basis. As a simple counterexample we can take two-dimensional space* $\mathbb{R}^2$ *with quadratic form* $q(x_1, x_2) = x_1^2 - x_2^2$, *and the bilinear form* $F(x, y) = x_1y_1 - x_2y_2$. *The vector* e_1 *with components* $(1, 1)$ *has the property* $q(e_1) = 0$, *but any vector orthogonal to this vector is automatically proportional to* e_1. *Thus* e_1 *can not be extended to an orthogonal basis.*

1.3.1 The center $Z(q)$ of $\mathrm{Cl}(q)$

For any algebra A its center $Z(A)$ is defined as the set of all these elements of the algebra that commute with every element of A

$$Z(A) = \{u \in A : ua = au \text{ for all } a \in A\}. \tag{49}$$

It follows from the definition that the center of any algebra A is a subalgebra of A, and that it always contains the scalar multiples of the identity of A. With the assumptions and notation as in Sec. 1.3 we will now find the center of the Clifford algebra $\mathrm{Cl}(q)$. First we will do it for a general, possibly degenerate q, then we will specialize to the case of nondegenerate q. Instead of stating the result first, and then providing a proof, we will take the opposite way: first we will discuss the subject and derive the result, and only then make it precise in the form of a proposition. We will use the fact that the algebra $\mathrm{Cl}(q)$ is graded into even and odd parts, cf. Eq. (20).

Suppose u is an element of the center and let us split it into the even and odd parts

$$u = u_0 + u_1,\ u_0 \in \mathrm{Cl}(q)^+, u_1 \in \mathrm{Cl}(q)^-. \tag{50}$$

Since u commutes with all elements of the algebra, it commutes, in particular, with all even elements $a_0 \in \mathrm{Cl}(q)^+$

$$(u_0 + u_1)a_0 = a_0(u_0 + u_1)$$

or

$$u_0 a_0 - a_0 u_0 = a_0 u_1 - u_1 a_0.$$

On the left we have even element, on the right - odd. Therefore both must be zero. Thus $u_0 a_0 = a_0 u_0$ and $a_0 u_1 = u_1 a_0$. We can do the same for odd elements a_1. The result is that if $u = u_0 + u_1$ is in the center, then the even part u_0 and the odd part u_1 are in the center. Therefore we can look separately for even and for odd elements of the center.

Let us first look for even elements u_0 in the center. We choose an orthogonal basis e_i in M, and the corresponding basis $e_{i_1}...e_{i_p}$, $(i_1 < i_2 < ... < i_p)$, in $C^p(q)$. For u to commute with all the elements of the algebra is the same as to commute with all elements of the basis $e_{i_1}...e_{i_p}$. We can also write u_0 as a linear combination of even elements, $e_{i_1}...e_{i_p}$, p even, of the basis of the algebra. Let us select the first vector e_1 of the basis e_i. We can then split u_0 into the part v_0 that is the linear combination of those $e_{i_1}...e_{i_p}$ that does not contain e_1, and the second part, made of those $e_{i_1}...e_{i_p}$ that contain e_1. Which we write as follows:

$$u_0 = v_0 + e_1 v_1, \tag{51}$$

where v_0 is even and does not contain e_1, and v_1 is odd and does not contain e_1. But now u_0 must commute with e_1, which means

$$e_1(v_0 + e_1 v_1) = (v_0 + e_1 v_1)e_1. \tag{52}$$

Since v_1 is odd, and since it does not contain e_1, it follows that v_1 anticommutes with e_1, i.e. $v_1 e_1 = -e_1 v_1$. Since v_0 is even and it does not contain e_1, it commutes with e_1. Therefore, from Eq. (52) we get that $e_1^2 v_1 = 0$. If $e_1^2 \neq 0$, which certainly happens if q is nondegenerate, we deduce that $v_1 = 0$. Therefore u_0 is even and does not contain e_1. The same we can repeat with e_2. We can move to the front in the expression $e_2 v_1$ changing the sign of v_1. The result is that u_0 does not contain in its expansion any element e_i with $e_i^2 \neq 0$.

We now investigate odd elements in the center. As before we write

$$u_1 = v_1 + e_1 v_0,$$

where v_1 is odd, v_0 is even, and neither v_1 nor v_0 does not contain e_1. This time e_1 commutes with v_0, therefore all we get from $u_1 e_1 = e_1 u_1$ is that v_1

commutes with e_1, which implies that $v_1 = 0$. Repeating this reasoning for e_2, e_3, etc. we conclude that u_1 is the proportional to the product $e_1....e_n$ of all basis elements. Since u_1 is odd, this can happen only if n is odd.

We summarise the above in the proposition below:

Proposition 1.33. *The even part of the center of* $\mathrm{Cl}(q)$ *consists of linear combinations of the even products of basis elements of* M *whose square is zero, and of the identity. The odd part of the center consists of the scalar multiples of the element* $e_1....e_n$ *if the dimension of* M *is odd, and consists of zero alone if the dimension of* M *is even.*

1.3.2 The algebras $\mathrm{Cl}_{p,q,r}$ in the real case (c.f. [18])

Let us now concentrate on the real case, when M is a real vector space of dimension n equipped with a (real-valued) quadratic form q. In that case if e_i is an orthogonal basis, $q(e_i)$ are real numbers. If $q(e_i)$ is positive, we will redefine e_i replacing it with $e_i \mapsto e_i/\sqrt{q(e_i)}$. For the new e_i we get $q(e_i) = +1$. If $q(e_i)$ is negative, we replace $e_i \mapsto e_i/\sqrt{-q(e_i)}$, and for the new e_i we get $q(e_i) = -1$. In this way we diagonalize the quadratic form q, so that on our basis vectors it has only values $+1$, -1, or 0. We now reorganize our basis so, that we have first basis vectors with square $+1$, say there are p of them, $e_1, ..., e_p$, then we have q basis vectors with square -1, $e_{p+1}, ..., e_{p+q}$, finally we have r basis vectors with square zero, $e_{p+q+1}, ..., e_{p+q+r}$, with $p+q+r = n$. We call such a basis *orthonormal.* The corresponding Clifford algebra is then denoted as $\mathrm{Cl}_{p,q,r}$. If $r = 0$, we simply write $\mathrm{Cl}_{p,q}$, and when $r = 0$ and $q = 0$, we write Cl_p.

We will use the notation $\mathrm{Cl}^0_{p,q,r}$ for the even subalgebra of $\mathrm{Cl}_{p,q,r}$.

We will now demonstrate several simple isomorphisms between Clifford algebras for different p and q.

Lemma 1.34. *For* $p \geq 1$ *we have the isomorphisms of algebras* $\mathrm{Cl}_{p,q,r} \simeq \mathrm{Cl}_{q+1,p-1,r}$.

Proof. Indeed, let $e_1, ..., e_p, e_{p+1}, ..., e_{p+q}, e_{p+q+1}, ..., e_{p+q+r}$ be an orthonormal basis for $\mathrm{Cl}_{p,q,r}$. we define a new basis $\tilde{e}_i$ as follows:

$$\tilde{e}_i = \begin{cases} e_1,\ i = 1; \\ e_i e_1,\ i = 2, ..., p+q+r. \end{cases} \tag{53}$$

We find that all $\tilde{e}_i$ anticommute with each other, that $\tilde{e}_1^2 = 1$, $\tilde{e}_2^2 = ... = \tilde{e}_p^2 = -1$, $\tilde{e}_{p+1}^2 = ... = \tilde{e}_{p+q}^2 = 1$, and $\tilde{e}_{p+q+1}^2 = ... = \tilde{e}_{p+q+r}^2 = 0$. Therefore the basis $\tilde{e}_i$ generates the Clifford algebra $\mathrm{Cl}_{q+1,p-1,r}$. Yet this is the same algebra as the original one. □

Remark 1.35. *The above isomorphism is the isomorphism of two algebras. That means there is a bijective linear map* $\phi : \mathrm{Cl}_{p,q,r} \to \mathrm{Cl}_{q+1,p-1,r}$ *such that* $\phi(ab) = \phi(a)\phi(b)$ *for all* $a, b \in \mathrm{Cl}_{p,q,r}$*. The two algebras are isomorphic as abstract algebras, identity is mapped into identity, but, for instance, their graded structures are not isomorphic. The mapping* ϕ *does not map odd elements into odd elements, also the main automorphism and anti-automorphism are different for the two algebras.*

Lemma 1.36. *For* $p \geq 4$ *we have the isomorphism of algebras* $\mathrm{Cl}_{p,q,r} \simeq \mathrm{Cl}_{p-4,q+4,r}$*.*

Proof. With the notation as in the proof of Lemma 1.34 we set

$$\tilde{e}_i = \begin{cases} e_i e_1 e_2 e_3 e_4, \ i = 1, 2, 3, 4; \\ e_i, \ i = 5, ..., n. \end{cases} \tag{54}$$

Notice that e_i anticommutes with $e_1 e_2 e_3 e_4$ for $i = 1, ..., 4$. Therefore, for $i \leq 4$, we have

$$\tilde{e}_i^2 = -(e_1 e_2 e_3 e_4)^2 = -1.$$

Therefore the first four vectors of the basis change it squares from $+1$ to -1. □

Remark 1.37. *In this case both algebras have the same even and odd parts. Therefore main automorphisms are the same. But main anti-automorphisms are not the same. Calculating the main anti-automorphism of the first algebra on* $\tilde{e}_1$ *we find*

$$\tilde{e}_1^\tau = (e_2 e_3 e_4)^\tau = e_4 e_3 e_2 = e_2 e_4 e_3 = -e_2 e_3 e_4 = -\tilde{e}_1$$

while the main anti-automorphism of the second algebra should leave $\tilde{e}_1$ *unchanged.*

Notice that in both lemmas the new generators $\tilde{e}_i$ *are linearly independent as they are proportional to the elements of the standard basis (without counting the identity* 1*)* $e_{i_1}...e_{i_k}$, $i_1 < .... < i_k$, $k = 1, ..., n$ *of the Clifford algebra .*

Lemma 1.38. *For* $q \geq 1$ *we have the isomorphism*

$$\mathrm{Cl}^0_{p,q,r} \simeq \mathrm{Cl}_{p,q-1,r}, \tag{55}$$

where $\mathrm{Cl}^0_{p,q,r}$ *denotes the even subalgebra of* $\mathrm{Cl}_{p,q,r}$*.*

Proof. Let e_i be an orthonormal basis with $e_i^2 = 1$, for $i = 1, ..., p$, $e_i^2 = -1$ for $i = p+1, ..., p+q$, and $e_i^2 = 0$ for $i = p+q+1, ..., n$. We define $\tilde{e}_i$, for $i = 1, ..., n$ as

$$\tilde{e}_i = e_i e_{p+q}. \tag{56}$$

We may skip $i = p+q$, since then $\tilde{e}_i = -1$. We obtain this way $n-1$ mutually anticommuting elements, with p squares $+1$, $q-1$ squares -1 and r squares zero. Therefore they generate the algebra $\mathrm{Cl}_{p,q-1,r}$. On the other hand they are all even elements of $\mathrm{Cl}_{p,q,r}$, and every even element of $\mathrm{Cl}_{p,q,r}$ can be obtained using $\tilde{e}_i$. Thus the lemma holds. □

1.3.2.1 Examples in low dimensions

It is important to know an explicit form of Clifford algebras in low dimensions, since then we can show the periodicity properties for finding their forms in higher dimensions.

It will be convenient to introduce the following four real matrices $\mathbf{1}, \iota, \theta, \kappa$:

$$\begin{aligned} \mathbf{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \iota = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \theta &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{57}$$

The three matrices ι, θ, κ mutually anticommute and together with the identity matrix $\mathbf{1}$ they form a basis in the 4-dimensional space $\mathrm{Mat}(2, \mathbb{R})$ of real 2×2 matrices. In fact these four matrices form an orthonormal basis with respect to the Euclidean scalar product in $\mathrm{Mat}(2, \mathbb{R})$ defined by $(u, v) = \frac{1}{2}\mathrm{Tr}(u^T v)$. We also notice that we have the following algebraic relations:

$$\iota\theta = \kappa, \ \kappa\theta = \iota, \ \kappa\iota = \theta. \tag{58}$$

1.3.2.1.1 $\mathrm{Cl}_0 \simeq \mathbb{R}$**.** Here $n = 0$, so the Clifford algebra is $2^0 = 1$-dimensional. The vector space is in this case zero-dimensional, it consists of the vector 0 alone. The Clifford algebra consists just of the scalar multiples of the identity.

1.3.2.1.2 $\mathrm{Cl}_1 \simeq \mathbb{R} \oplus \mathbb{R}$**.** Here $n = 1$ and the Clifford algebra is 2-dimensional. Apart of the identity there is just one basis vector with square 1. It is convenient to represent such a direct sum as block diagonal matrices, in this case with real numbers on the diagonal. Using the notation of Eq. (57) we can choose:

$$e_1 \mapsto \theta. \tag{59}$$

Remark 1.39. *We could also choose* $e_1 \mapsto \kappa$*. To see that choosing* $e_1 = \theta$ *is better, we notice that defining*

$$e_+ = \frac{1}{2}(1 + e_1),\ e_- = \frac{1}{2}(1 - e_1) \tag{60}$$

we have $e_+ + e_- = 1$*,* $e_+e_- = e_-e_+ = 0$*,* $(e_+)^2 = e_+$*,* $(e_-)^2 = e_-$*. In other words* e_+ *and* e_- *are two orthogonal projections, and the sets* $\{\alpha e_+ : \alpha \in \mathbb{R}\}$ *and* $\{\beta e_- : \beta \in \mathbb{R}\}$ *are two algebras, each one isomorphic to* $\mathbb{R}$*. When* $e_1 = \theta$ *these two projections are represented by matrices*

$$e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{61}$$

and their properties are evident from the matrix form. If we choose $e_1 = \kappa$*, then*

$$e_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ e_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{62}$$

which are rather clumsy. Moreover, for $e_1 = \theta$ *it is evident that the whole algebra is represented by diagonal matrices* $\alpha e_+ + \beta e_-$

$$\alpha e_+ + \beta e_- = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \tag{63}$$

while for $e_1 = \kappa$ *the whole algebra is represented by matrices of the form*

$$\frac{1}{2}\begin{pmatrix} \alpha + \beta & \alpha - \beta \\ \alpha - \beta & \alpha + \beta \end{pmatrix} \tag{64}$$

in which separating the two algebras is more involved.

1.3.2.1.3 $\mathrm{Cl}_{0,1} \simeq \mathbb{C}$**.** The algebra is spanned by the identity and one basis vector with square minus one $e_1^2 = -1$. Using the notation of Eq. (57) we can represent the generator by the matrix:

$$e_1 \mapsto \iota. \tag{65}$$

Identifying e_1 with i, the imaginary square root of -1, the algebra becomes isomorphic to complex numbers $\mathbb{C}$.

1.3.2.1.4 $\mathrm{Cl}_{0,0,1}$ **- the Dual numbers** Here we have one basis vector with square 0. We can represent the generators by the matrices:

$$\mathbf{1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{66}$$

The algebra is known under the name of *dual numbers*.

1.3.2.1.5 $\mathrm{Cl}_{0,2} \simeq \mathbb{H}$**.** The quadratic form[3] $q(x)$ is in this case $q(x^1, x^2) = -(x^1)^2 - (x^2)^2$. We have two anticommuting generators e_1, e_2 with squares -1

$$e_1^2 = e_2^2 = -1,\ e_1e_2 = -e_2e_1. \tag{67}$$

The elements $1, e_1, e_2, e_{12} = e_1e_2$ form the basis of the algebra. We find that e_{12} has also square -1 and it anticommutes with e_1 and e_2. Using the substitution

$$e_1 \to \mathrm{i}, \qquad e_2 \to \mathrm{j}, \qquad e_{12} \to \mathrm{k}$$

where i, j, and k are imaginary units of quaternions, we obtain the isomorphism $\mathrm{Cl}_{0,2} \simeq \mathbb{H}$ - the algebra of quaternions. While the algebra of quaternions is isomorphic to the Clifford algebra $\mathrm{Cl}_{0,2}$ the isomorphism is not a *natural* one. The natural function of quaternions is to implement rotations in $\mathbb{R}^3$. In order to understand better the role of quaternions as representing elements of $\mathrm{Cl}_{0,2}$ let us find how are the Clifford algebra operations such as trace, main involution and main anti-involution represented in $\mathbb{H}$.

Every quaternion q is written as $q = \alpha_0 + \alpha_1\mathrm{i} + \alpha_2\mathrm{j} + \alpha_3\mathrm{k}$, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are real numbers. This corresponds to the element q of the Clifford algebra $\mathrm{Cl}_{0,2}$

$$q = \alpha_0 + \alpha_1e_1 + \alpha_2e_2 + \alpha_{12}e_{12}. \tag{68}$$

Therefore the trace of q, as it is defined in Sec. 1.4.2, is simply the scalar part α_0 of the quaternion. The main involution α is an automorphism of the algebra that changes the signs of odd vectors. In our case it should change the sign of i and j, but not of k. It is easy to guess its form acting on quaternions[4]:

$$\alpha(q) = \mathrm{k}q\mathrm{k}^{-1}. \tag{69}$$

Anti-automorphism τ should change the order of multiplication, but should not change the signs of e_1 and e_2. Quaternions have a well known anti-automorphism, the conjugation $q \mapsto q^*$ which changes the sign of the imaginary units $\mathrm{i}, \mathrm{j}, \mathrm{k}$. To make it not to change the signs of e_1, e_2 we must combine it with the previous automorphism. Thus:

$$\tau(q) = \mathrm{k}\, q^*\, \mathrm{k}^{-1}. \tag{70}$$

We can represent the algebra in two-dimensional complex vector space as follows:

[3]The symbol q for the quadratic form should not be confused with the symbol q used for a generic quaternion.

[4]Where of course $\mathrm{k}^{-1} = -\mathrm{k}$.

$$e_1 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{71}$$

Then

$$e_1e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{72}$$

and the whole algebra consists of matrices of the form:

$$u = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} = \{\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} : z_j \in \mathbb{C}\}. \tag{73}$$

1.3.2.1.6 $\mathrm{Cl}_3 \cong \mathrm{Mat}(2, \mathbb{C})$. The algebra is $2^3 = 8$-dimensional. We have three anticommuting generators with squares 1. They can be represented by the Pauli matrices:

$$1 \mapsto \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \mapsto \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$e_2 \mapsto \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad e_3 \mapsto \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{74}$$

We have:

$$\sigma_1\sigma_2 = \mathrm{i}\sigma_3, \quad \sigma_2\sigma_3 = \mathrm{i}\sigma_1, \quad \sigma_3\sigma_1 = \mathrm{i}\sigma_2, \quad \sigma_1\sigma_2\sigma_3 = \mathrm{i}.$$

Notice that while $\sigma_1\sigma_2$ is proportional to σ_3 with the complex proportionality constant, it is independent of σ_3 as an element of the real vector space. The eight complex matrices $\sigma_0, \sigma_i, \sigma_{ij}$ $(i < j)$, and $\sigma_1\sigma_2\sigma_3 = \mathrm{i}$ form a real basis in the space of 2×2 of complex matrices. Every complex matrix 2×2 can be written as a linear combination of these eight matrices with real coefficients. The space of 2×2 complex matrices has 4 complex dimensions, that is 8 real dimensions.

As we did it with quaternions, so here we will identify the trace, the main automorphism, and the main anti-automorphism of the Clifford algebra Cl_3 realized as the algebra of all complex 2×2 matrices.

The trace is easy, it should be the real coefficient in front of the identity matrix. So it must be $1/2$ of the real part of the ordinary trace of the matrix.

We now consider the main automorphism. It should change the sign of the three Pauli matrices. Matrices σ_1 and σ_3 are real, while σ_2 is imaginary. The formula that works can be obtained after some little work. For a complex 2×2 matrix a we find that[5]

$$\alpha(u) = \sigma_2 \bar{u} \sigma_2^{-1},$$

[5]Where, of course, $\sigma_2^{-1} = \sigma_2$.

where $\bar{a}$ denotes the complex conjugated matrix. Explicitely:

$$\alpha : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}. \tag{75}$$

The main anti-automorphism should reverse the order of multiplication, but must leave the Pauli matrices unchanged. All three Pauli matrices are Hermitian, therefore the Hermitian conjugate of the complex matrices (complex conjugate transpose, $a \mapsto a^* = \bar{a}^t$) does the job:

$$\tau(a) = a^*. \tag{76}$$

The composition of τ and α is sometimes called *the conjugation*, and it is denoted $nu(u)$. Explicitely, if

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{77}$$

then

$$u^\nu = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{78}$$

We also have the following useful property:

$$uu^\nu = u^\nu u = \det(u)\mathbf{1}. \tag{79}$$

1.3.2.1.7 $\mathrm{Cl}_{1,1} \simeq \mathrm{Mat}(2, \mathbb{R})$**.** The quadratic form is now $q(x^1, x^2) = (x^1)^2 - (x^2)^2$. The Clifford algebra is $2^2 = 4$-dimensional. We can represented it as the algebra of all real 2×2 matrices (which is also $2 \times 2 = 4$-dimensional) by defining generators e_1, e_2 with squares 1 and -1 as follows:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \, e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \, e_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{80}$$

The element e_1e_2 is now represented by the matrix

$$e_1e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{81}$$

These four matrices span the whole algebra of 2×2 real matrices. The trace in the Clifford algebra is now $1/2$ of the trace of the matrix. The main automorphism is realized as $a \mapsto e_{12} a e_{12}^{-1}$, the main anti-automorphism as $a \mapsto e_1 a^t e_1^{-1}$.

1.3.2.1.8 $\mathrm{Cl}_{1,2} \simeq \mathrm{Mat}(2,\mathbb{C})$ We need three anticommuting matrices, one with square 1, and two with square -1. We can use to this end Pauli matrices, multiplied by imaginary i to get square -1. There is a freedom of choice here, let us choose the following representation:

$$\begin{aligned} e_1 \mapsto \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e_2 \mapsto i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ e_3 \mapsto -i\sigma_2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{82}$$

With this choice we have

$$e_1 e_2 e_3 \mapsto i\mathbf{1}, \tag{83}$$

therefore we have at our disposal all complex numbers, and Pauli matrices - they generate the whole algebra $\mathrm{Mat}(2,\mathbb{C})$ of 2×2 complex matrices. The Clifford algebra $\mathrm{Cl}_{2,1}$ has $2^3 = 4 \times 2$ real dimensions.

In order to find the main automorphism we notice that we have at our disposal the complex conjugation operation. It reverses the sign of e_2, but leaves e_1 and e_3 invariant. Therefore we add conjugation by σ_1 to obtain

$$\alpha(u) = \sigma_1 \bar{u} \sigma_1^{-1}. \tag{84}$$

In order to find the main anti-automorphism we notice that we have at our disposal the Hermitian conjugation operation $u \mapsto u^* = \bar{u}^t$. It reverses the signs of e_2 and e_3. Therefore we combine it with the conjugation by σ_3 to obtain

$$\tau(u) = \sigma_3 u^* \sigma_3^{-1}. \tag{85}$$

It is easy to find the explicit form of the conjugation $\nu(u) = \alpha(\tau(u))$

$$u \mapsto \nu(u) = \alpha(\tau(u)) = \tau(\alpha(u)) = C u^t C^{-1}, \tag{86}$$

where

$$C = e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{87}$$

Explicitely, if

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{88}$$

then

$$u^\nu = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{89}$$

which has the same form as in Eq. (78).

1.3.2.1.9 $\mathrm{Cl}_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R})$ - Majorana representation

The quadratic form for the Minkowski space of Special Relativity can be either of signature $(1,3)$ or $(3,1)$. Here we consider the signature $(3,1)$. We choose orthonormal basis e_i,$(i=1,...,4)$ with $q(e_1)=q(e_2)=q(e_3)=1$, and $q(e_4)=-1$. Using the notation of Eq. (57) the Clifford algebra $\mathrm{Cl}_{3,1}$ can then be generated by 4×4 real matrices in a block matrix form as follows:

$$\begin{aligned} e_1 &= \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & \iota \\ -\iota & 0 \end{pmatrix}, \\ e_3 &= \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix}. \end{aligned} \tag{90}$$

The matrices satisfy the anticommutation relations[6]

$$e_ie_j + e_je_i = 2\eta_{ij}, \tag{91}$$

where

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{92}$$

The matrices e_i generate the whole $2^4 = 16$-dimensional Clifford algebra. Apart of the identity and the four matrices e_i, we have six matrices e_ie_j, $(i<j)$, four matrices $e_ie_je_k$, $(i<j<k)$, and one matrix $\omega = e_1e_2e_3e_4$. This last matrix anticommutes with the matrices e_i, and has square -1:

$$\omega = e_1e_2e_3e_4 = \begin{pmatrix} -\iota & 0 \\ 0 & \iota \end{pmatrix}. \tag{93}$$

The sixteen real matrices so obtained span the whole algebra $\mathrm{Mat}(4,\mathbb{R})$ of real 4×4 matrices. Of course the representation given in Eq. (90) is not a unique one. Given any invertible 4×4 real matrix S, the matrices $\tilde{e}_i = Se_iS^{-1}$ provide another possible representation. In fact, if we have any four real matrices $\tilde{e}_i$ satisfying the same anticommutation relations as e_i (cf. Eq. 91)), then there exists real invertible matrix S, unique up to a non-zero real multiplier, such that $\tilde{e}_i = Se_iS^{-1}$. Representation of the Clifford algebra $\mathrm{Cl}_{3,1}$ by real 4×4 matrices is often called *the Majorana representation.* Because the matrix ω anticommutes with all e_i, $(i=1,...,4)$ we can choose it to implement the main automorphism α:

$$\alpha(u) = \omega u \omega^{-1},\ u \in \mathrm{Cl}_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R}). \tag{94}$$

[6] To be precise, on the right hand side of Eq. (91) we should put $\eta_{ij}\mathbf{1}_4$, where $\mathbf{1}_4$ is the 4×4 identity matrix.

To implement the main anti-automorphism we notice the matrices representing e_1, e_2, e_3 are symmetric, while e_4 is antisymmetric. The transposition, which is an anti-automorphism of the matrix algebra $\mathrm{Mat}(4,\mathbb{R})$ would leave e_1, e_2, e_3 invariant, but it will change the sign of e_4. Therefore we introduce the matrix $T = e_1e_2e_3$:

$$T = e_1e_2e_3 = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{95}$$

which commutes with e_1, e_2, e_3 and anticommutes with e_4. We can now implement the main anti-involution τ as

$$\tau(u) = T\, u^t\, T^{-1}, \tag{96}$$

where $u \mapsto u^t$ stands for the transposition of matrices.

1.3.2.1.10 $\mathrm{Cl}_{1,3} \simeq \mathrm{Mat}(2,\mathbb{H})$ - Chiral (Weyl) and Dirac representations

Here M is the Minkowski space, and we will use coordinates $x = (x_0, x_1, x_2, x_3)$ with the quadratic form $q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$. Since we will use several different matrix representation of the Clifford algebra $\mathrm{Cl}_{1,3}$, we will use different symbols for matrices representing the orthonormal basis e_i, $(i = 0, .., 3)$. The simplest representation is by 2×2 matrices with quaternion entries:

$$g_0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},\; g_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix},\; g_2 = \begin{pmatrix} \mathrm{j} & 0 \\ 0 & -\mathrm{j} \end{pmatrix},\; g_3 = \begin{pmatrix} \mathrm{k} & 0 \\ 0 & -\mathrm{k} \end{pmatrix}. \tag{97}$$

It is evident that the matrices satisfy the necessary anticommution relations. They also generate the whole $2^4 = 4 \times 4$-dimensional algebra $\mathrm{Mat}(2,\mathbb{H})$. Here 2^4 is the dimension of the Clifford algebra $\mathrm{Cl}_{1,3}$, and 4×4 is the dimension of the algebra $\mathrm{Mat}(2,\mathbb{H})$. We can now implement the representation of quaternions by complex gamma matrices as in Eqs (71),(72) to obtain the following representation in terms of 4×4 complex matrices:

$$\begin{aligned} g_0 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\; g_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \\ g_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\; \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}. \end{aligned} \tag{98}$$

The matrix representation above shows relation to the $\mathrm{Mat}(2,\mathbb{H})$ algebra, but physicists routinely use different representations. One of them is called *chiral* or *Weyl* representation. It is defined by the following matrices Γ_i:

$$\begin{aligned}\Gamma_0 &= \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\ \Gamma_1 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\\ \Gamma_2 &= \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix},\ \Gamma_3 = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}.\end{aligned} \tag{99}$$

The two representations g_i and Γ_i are *equivalent*, that is there exists invertible 4×4 complex matrix S such that

$$Sg_iS^{-1} = \Gamma_i,\ (i = 0, ..., 3). \tag{100}$$

Explicitely:[7]

$$S = \frac{1}{2}\begin{pmatrix}1&-1&-i&i\\-1&-1&i&i\\-i&i&1&-1\\i&i&-1&-1\end{pmatrix}. \tag{101}$$

The matrices Γ_i are all block antidiagonal. The *Dirac* representation uses matrices γ_i with $\gamma_i = \Gamma_i$ for $i = 1, 2, 3$, but

$$\gamma_0 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{102}$$

Again the Dirac representation is equivalent to the Weyl representation:

$$\gamma_i = S_1\Gamma_iS_1^{-1},\ (i = 0, ..., 3), \tag{103}$$

with[8]

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}. \tag{104}$$

[7] In practice it is simpler to verify the equations $Sg_i = \Gamma_iS$. This avoids calculating S^{-1} even though the determinant of the matrix S given above is equal to 1.

[8] the square root of 2 is there just to make the determinant of S_1 equal to 1.

1.3.2.1.11 $\mathrm{Cl}_{p+1,q+1} \simeq \mathrm{Mat}(2, \mathrm{Cl}_{p,q})$. In general the notation $\mathrm{Mat}(2, A)$ denotes the algebra of 2×2 matrices the entries of which are elements of the algebra A. Let e_i be the basis of the $n = p + q$ dimensional space generating the 2^n-dimensional Clifford algebra $\mathrm{Cl}_{p,q}$. Let $\mathbf{1}$ be the identity element of this algebra. We define $n + 2$ generators of the algebra $\mathrm{Cl}_{p+1,q+1}$ as follows

$$e_i \mapsto \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix},\ e^+ \mapsto \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},\ e^- \mapsto \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{105}$$

The matrices on the right are 2×2 block matrices, with blocks of the size $2^n \times 2^n$. They form $4 \times 2^n = 2^{n+2}$ algebra.

Of course the isomorphism $\mathrm{Cl}_{1,1} \simeq \mathrm{Mat}(2, \mathbb{R})$ is a particular case of the above general isomorphism, for $n = 0$.

1.3.2.2 The table of real Clifford algebras $\mathrm{Cl}_{p,q}$ The following classification of all real Clifford algebras $\mathrm{Cl}_{p,q}$ can be obtained by following the reasoning like those above (c.f. [18] and references therein):

Theorem 1.40 (Cartan 1908)**.** *We have the following isomorphism of algebras*

$$\mathrm{Cl}_{p,q} \cong \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{R}), & \text{if } p - q \equiv 0; 2 \mod 8 \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{R}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{R}), & \text{if } p - q \equiv 1 \mod 8 \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}), & \text{if } p - q \equiv 3; 7 \mod 8 \\ \mathrm{Mat}(2^{\frac{n-2}{2}}, \mathbb{H}), & \text{if } p - q \equiv 4; 6 \mod 8 \\ \mathrm{Mat}(2^{\frac{n-3}{2}}, \mathbb{H}) \oplus \mathrm{Mat}(2^{\frac{n-3}{2}}, \mathbb{H}), & \text{if } p - q \equiv 5 \mod 8. \end{cases}$$

The form of the matrix representation of the algebra shows a specific periodicity with respect to $d = p - q \mod 8$. Table 1 shows all Clifford algebras $\mathrm{Cl}_{p,q}$ for $n = p + q$ from 0 to 12. We notice that for n even they are always isomorphic to full matrix algebras with entries being real, complex or quaternionic. In each case they are being considered as real algebras, so that a complex number is considered to be a pair of real numbers, and a quaternion is considered to be four real numbers.

The important element of each Clifford algebra $\mathrm{Cl}_{p,q}$ is its *volume element*, let us denote it as ω. If e_i is an orthonormal basis, then

$$\omega = e_1 e_2 ... e_n. \tag{106}$$

For n even the volume element anticommutes with all basis vectors e_i. For n odd it always commutes - we know that it spans the center of the algebra (cf. Sec. 1.3.1). In that case it is very important whether its square is $+1$ or

-1. Lets us calculate $\omega^2 = e_1...e_n\, e_1...e_n$. We have to commute e_1 that occurs after e_n through e_n,..,e_2, until we get $(e_1)^2$ at the beginning. Each time we change the sign, because $e_ie_j = -e_je_i$ for $i \neq j$. Thus we will change the sign $n-1$ times. Then we have to do the same with e_2. This will change the sign $n-2$ times. And so on, until we get $(e_1)^2....(e_n)^2$. Altogether we will change the sign $(n-1)+(n-2)+,,,+1 = n(n-1)/2$ times. On the other hand $e_1^2....e_n^2 = (+1)^p(-1)^q = (-1)^q$. Therefore we obtain:

$$\omega^2 = (-1)^{n(n-1)/2+q}. \tag{107}$$

Now, we have $p+q = n$, $p-q = d$, therefore $(n(n-1)/2+q = (n^2-d)/2$, and so

$$\omega^2 = (-1)^{\frac{1}{2}(n^2-d)}. \tag{108}$$

If n is odd then $n = 2k+1$,, therefore $n^2 = 4k^2+4k+1 = 4(k^2+k)+1$, therefore $(-1)^{n^2} = -1$ and so

$$\omega^2 = (-1)^{\frac{d+1}{2}}. \tag{109}$$

If n is odd, then also d is odd. It is clear from the last formula that ω^2 retains the sign when d increases by 4. When $d = 1 \mod 4$ we have $\omega^2 = 1$, when $d = 3 \mod 4$, we have $\omega^2 = -1$. We have thus showed that the following property holds:

Proposition 1.41. *For n odd we have that*

$$\omega^2 \doteq (e_1...e_n)^2 = \begin{cases} 1, & \text{if } p-q = 1 \mod 4 \\ -1, & \text{if } p-q = 3 \mod 4 \end{cases} \tag{110}$$

1.3.2.2.1 The case of $p+q$ odd and $p-q = 1 \mod 4$ (cf. [14, p. 22]) This is the case when $\omega = e_1...e_n$ commutes with all the elements of the algebra. Since ω is odd we have $\alpha(\omega) = -\omega$, where α is the main automorphism (involution) of the algebra (cf. Sec. 1.2.2). Let us introduce π^+, π^- as follows:

$$\pi^{\pm} = \frac{1}{2}(1 \pm \omega). \tag{111}$$

Then $\pi^{\pm}$ are idempotents with sum equal 1:

$$(\pi^{\pm})^2 = \pi^{\pm}, \quad \pi^+ + \pi^- = 1, \quad \pi^+\pi^- = \pi^-\pi^+ = 0. \tag{112}$$

Moreover they commute with every element of the algebra, and we have

$$\alpha(\pi^{\pm}) = \pi^{\mp}. \tag{113}$$

Therefore each element a can be split into two parts $a = \pi^+ a + \pi^- a$, and the whole algebra can be split into two ideals

$$\mathrm{Cl}_{p,q} = \mathrm{Cl}^+_{p,q} \oplus \mathrm{Cl}^-_{p,q}, \tag{114}$$

where

$$\mathrm{Cl}^\pm_{p,q} = \pi^\pm \mathrm{Cl}_{p,q} = \mathrm{Cl}_{p,q}\pi^\pm = \pi^\pm \mathrm{Cl}_{p,q}\pi^\pm. \tag{115}$$

Moreover the two ideals are isomorphic to each other:

$$\alpha(\pi^\pm) = \pi^\mp. \tag{116}$$

The above reasoning explains why in Table 1, in every row with odd n and $d = 1 \mod 4$ we have entries of the form ${}^2X$, which is a short notation for $X \oplus \mathrm{X}$, where X is one of the full matrix algebras.

$n\backslash d$	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
0	·	·	·	·	·	·	·	·	·	·	·	·	$\mathbb{R}$	·	·	·	·	·	·	·	·	·	·	·	·
1	·	·	·	·	·	·	·	·	·	·	·	$\mathbb{C}$	·	${}^2\mathbb{R}$	·	·	·	·	·	·	·	·	·	·	·
2	·	·	·	·	·	·	·	·	·	·	$\mathbb{H}$	·	$\mathbb{R}(2)$	·	$\mathbb{R}(2)$	·	·	·	·	·	·	·	·	·	·
3	·	·	·	·	·	·	·	·	·	${}^2\mathbb{H}$	·	$\mathbb{C}(2)$	·	${}^2\mathbb{R}(2)$	·	$\mathbb{C}(2)$	·	·	·	·	·	·	·	·	·
4	·	·	·	·	·	·	·	·	$\mathbb{H}(2)$	·	$\mathbb{H}(2)$	·	$\mathbb{R}(4)$	·	$\mathbb{R}(4)$	·	$\mathbb{H}(2)$	·	·	·	·	·	·	·	·
5	·	·	·	·	·	·	·	$\mathbb{C}(4)$	·	${}^2\mathbb{H}(2)$	·	$\mathbb{C}(4)$	·	${}^2\mathbb{R}(4)$	·	$\mathbb{C}(4)$	·	${}^2\mathbb{H}(2)$	·	·	·	·	·	·	·
6	·	·	·	·	·	·	$\mathbb{R}(8)$	·	$\mathbb{H}(4)$	·	$\mathbb{H}(4)$	·	$\mathbb{R}(8)$	·	$\mathbb{R}(8)$	·	$\mathbb{H}(4)$	·	$\mathbb{H}(4)$	·	·	·	·	·	·
7	·	·	·	·	·	${}^2\mathbb{R}(8)$	·	$\mathbb{C}(8)$	·	${}^2\mathbb{H}(4)$	·	$\mathbb{C}(8)$	·	${}^2\mathbb{R}(8)$	·	$\mathbb{C}(8)$	·	${}^2\mathbb{H}(4)$	·	$\mathbb{C}(8)$	·	·	·	·	·
8	·	·	·	·	$\mathbb{R}(16)$	·	$\mathbb{R}(16)$	·	$\mathbb{H}(8)$	·	$\mathbb{H}(8)$	·	$\mathbb{R}(16)$	·	$\mathbb{R}(16)$	·	$\mathbb{H}(8)$	·	$\mathbb{H}(8)$	·	$\mathbb{R}(16)$	·	·	·	·
9	·	·	·	${}^2\mathbb{C}(16)$	·	${}^2\mathbb{R}(16)$	·	$\mathbb{C}(16)$	·	${}^2\mathbb{H}(8)$	·	$\mathbb{C}(16)$	·	${}^2\mathbb{R}(16)$	·	$\mathbb{C}(16)$	·	${}^2\mathbb{H}(8)$	·	$\mathbb{C}(16)$	·	${}^2\mathbb{R}(16)$	·	·	·
10	·	·	$\mathbb{H}(16)$	·	${}^2\mathbb{R}(32)$	·	$\mathbb{R}(32)$	·	${}^2\mathbb{H}(16)$	·	$\mathbb{H}(16)$	·	${}^2\mathbb{R}(32)$	·	$\mathbb{R}(32)$	·	${}^2\mathbb{H}(16)$	·	$\mathbb{H}(16)$	·	$\mathbb{R}(32)$	·	$\mathbb{R}(32)$	·	·
11	·	${}^2\mathbb{H}(16)$	·	$\mathbb{C}(32)$	·	${}^2\mathbb{R}(32)$	·	$\mathbb{C}(32)$	·	${}^2\mathbb{H}(16)$	·	$\mathbb{C}(32)$	·	${}^2\mathbb{R}(32)$	·	$\mathbb{C}(32)$	·	${}^2\mathbb{H}(16)$	·	$\mathbb{C}(32)$	·	${}^2\mathbb{R}(32)$	·	$\mathbb{C}(32)$	·
12	$\mathbb{H}(32)$	·	$\mathbb{H}(32)$	·	$\mathbb{R}(64)$	·	$\mathbb{R}(64)$	·	$\mathbb{H}(32)$	·	$\mathbb{H}(32)$	·	$\mathbb{R}(64)$	·	$\mathbb{R}(64)$	·	$\mathbb{H}(32)$	·	$\mathbb{H}(32)$	·	$\mathbb{R}(64)$	·	$\mathbb{R}(64)$	·	$\mathbb{H}(32)$

Table 1: Isomorhisms between $\mathrm{Cl}_{p,q}$ and matrix algebras. Here $d = p - q$. Here, for instance, $\mathbb{H}(64)$ denotes the algebra $\mathrm{Mat}(64, \mathbb{H})$, and ${}^2\mathbb{R}(32)$ denotes the direct sum $\mathrm{Mat}(32, \mathbb{R}) \oplus \mathrm{Mat}(32, \mathbb{R})$. The table has a left-right symmetry (as much as possible) with respect to the vertical $d = 1$ line. The even subalgebra is always North-East of a given entry (cf. Lemma 1.38.)

1.4 Complex Clifford algebras

In physics Clifford algebras are usually represented as matrix algebras. These matrices act on vectors, and these vectors usually represent quantum states of particles with spin, often they are called *spinors*. For reasons that are still not completely understood quantum theory is always using complex numbers. Therefore matrices act on vectors with complex components. If we have some Clifford algebra $\mathrm{Cl}_{p,q}$ represented by matrices acting on a complex vector space, and if we choose an orthonormal basis e_i in M, we will have p matrices with square $+1$ and q matrices with square -1. But then, since the space on which these matrices are acting is complex, we can replace the matrices e_i with square -1 by ie_i, where i is the complex imaginary unit. In this way we will have representation of the Clifford algebra $\mathrm{Cl}_{n,0}$.

The above can be also considered in a more formal way. Given a real Clifford algebra $\mathrm{Cl}_{p,q}$ we can *complexify* it defining $\mathrm{Cl}^{\mathbb{C}}_{p,q} = \mathbb{C} \otimes \mathrm{Cl}_{p,q}$. Taking tensor product with the field $\mathbb{C}$ of complex numbers means that we extend our algebra by enlarging the field of scalars. Every element u of the complexified algebra is now a pair (v, w) of elements of the real algebra, interpreted as $u = v + iw$.

Alternatively, we can start with complexifying M by constructing $M^{\mathbb{C}} = \mathbb{C} \otimes M = M \oplus iM$, and extending by linearity the real bilinear form $F(x, y)$ to complex valued bilinear form $F^{\mathbb{C}}$, we the have the complex valued quadratic form $q^{\mathbb{C}}(x) = F^{\mathbb{C}}(x, x)$. The Clifford algebra of the complexified space for the complexified form is the same as the complexified real Clifford algebra discussed above.

In the complexified space $M^{\mathbb{C}}$ we can always find an orthonormal basis e_i for $F^{\mathbb{C}}$ with $F^{\mathbb{C}}(e_i, e_j) = \delta_{ij}$ with squares of the basis vectors always being $+1$. In other words, in the complex domain there is no sense to consider Clifford algebras $\mathrm{Cl}^{\mathbb{C}}_{p,q}$. We are discussing only Clifford algebras $\mathrm{Cl}^{\mathbb{C}}_{n}$.

1.4.1 Matrix representation of the Clifford algebras $\mathrm{Cl}^{\mathbb{C}}_{n}$.

We can construct Clifford algebras $\mathrm{Cl}^{\mathbb{C}}_{n}$ recursively (cf. [19]). We will see that it is important whether n is odd or even, and that is the only property of n that counts if we are interested in the form of the algebra. We start with $n = 1$. So n is odd and the Clifford algebra, as a complex vector space, has dimension $2^1 = 2$. It is spanned by two 2×2 matrices:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{117}$$

Thus the whole algebra consists of matrices

$$u = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where a, b are complex numbers. Using the notation of Table 1 we have $\mathrm{Cl}_1^{\mathbb{C}} = {}^2\mathbb{C}$. For $n = 2$, n is even, and the Clifford algebra has complex dimension $2^2 = 4$ and we set

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{118}$$

Then

$$e_1 e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{119}$$

and the four matrices $\mathbf{1}, e_1, e_2, e_1e_2$ span the whole algebra $\mathrm{Mat}(2, \mathbb{C})$ of complex 2×2 matrices.

We now give the recursive formula. First we give the formula how to construct the Clifford algebra for odd $n = 2k + 1$ if we have already constructed the Clifford algebra for $n = 2k$.

Suppose we have constructed matrices $e_1, ..., e_n$ for the even $n = 2k$ Clifford algebra $\mathrm{Cl}_n^{\mathbb{C}}$. Then we construct the matrices generating the next $\mathrm{Cl}_{2k+1}^{\mathbb{C}}$ using the formula

$$e_i \mapsto \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix}, \quad (i = 1, ..., 2k), \tag{120}$$

$$e_{2k+1} \mapsto \begin{pmatrix} i^k \omega_n & 0 \\ 0 & -i^k \omega_n \end{pmatrix}, \tag{121}$$

where

$$\omega_n = e_1 ... e_n. \tag{122}$$

We first notice that e_{2k+1} anticommutes with all e_i for $= 1, ..., k$. This follows from the fact that e_i anticommute with $e_1...e_n$. Indeed, in $e_1...e_n$ we have odd number of e_j different from a given e_i. Then we notice that $e_{2k+1}^2 = 1$. This follows directly form the formula (107). In our case, for $n = 2k$, it reads:

$$\omega^2 = (-1)^{n(n-1)/2} = (-1)^k. \tag{123}$$

Therefore

$$e_{2k+1}^2 = \begin{pmatrix} i^{2k} \omega_n^2 & 0 \\ 0 & (-i)^{2k} \omega_n^2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{124}$$

The matrices above are block matrices. Notice that the size of the new matrices, for odd $n = 2k+1$, is twice the size of the matrices of the preceding Clifford algebra for $n = 2k$. But all the matrices for $n = 2k+1$ are block diagonal.

Now we give the formula for constructing the Clifford algebra $\mathrm{Cl}^{\mathbb{C}}_{2k+2}$ if we have already constructed the matrices representing the Clifford algebra Cl_{2k+1}. In this step the size of the matrices is not increasing. The matrices e_i for $i = 1, ..., 2k+1$ remain the same, and we define the new matrix e_{2k+2} as

$$e_{2k+2} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{125}$$

Now we have to show that the new matrix anticommutes with all the previous ones. But this follows immediately from the form of e_{2k+2} and the fact that the other e_i-s are block diagonal of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & -A \end{smallmatrix}\right)$.

Remark 1.42. *The recursive construction given above is one of the many possible. For instance in Ref. [20, Sec. 4.2.1] Trautman gives a different recursive prescription that results in real matrices, but Trautman's formula instead of leading to* $\mathrm{Cl}^{\mathbb{C}}_{n,0}$*, leads to* $\mathrm{Cl}^{\mathbb{C}}_{m+1,m}$ *or* $\mathrm{Cl}^{\mathbb{C}}_{m,m}$ *Clifford algebras. Their generators can then be converted to generators of* $\mathrm{Cl}^{\mathbb{C}}_{n,0}$ *using the multiplication by* i *of those with square* -1..

Theorem 1.43. *If* n *is even,* $n = 2m$ *then* $\mathrm{Cl}^{\mathbb{C}}_n$ *coincides with the full algebra* $\mathrm{Mat}(2^m, \mathbb{C})$ *of complex* $2^m \times 2^m$ *matrices. If* n *is odd,* $n = 2m+1$*, then* $\mathrm{Cl}^{\mathbb{C}}_n$ *coincides with the algebra* ${}^2\mathrm{Mat}(2^m, \mathbb{C})$ *of block-diagonal* $2 \cdot 2^2 m \times 2 \cdot 2^m$ *matrices*

Proof. The (complex) dimension of $\mathrm{Cl}^{\mathbb{C}}_n$ is 2^n. For n odd our recursive construction above leads to block diagonal matrices. Matrices $2^m \times 2^m$ form algebra of dimension $2^m \times 2^m = 2^{2m}$ and block diagonal matrices with blocks of that dimension form the algebra of dimension $2^{2m+1} = 2^n$. Therefore the two algebras coincide. Similar reasoning applies to the case of n even. We know $\mathrm{Cl}^{\mathbb{C}}_n$ is represented by matrices $2^m \times 2^m$, but because of the dimensional reasons $\mathrm{Cl}^{\mathbb{C}}_n$ must be the full algebra $\mathrm{Mat}(2^m, \mathbb{C})$.

1.4.2 The trace and the bilinear form on $\mathrm{Cl}(q)$

With the assumptions and the notation as above we have the direct sum decomposition

$$\mathrm{Cl}(q) = \bigoplus_{p=0}^{n} \mathrm{Cl}(q)^p. \tag{126}$$

We denote by Σ the set of all 2^n ordered sequences $i_1 < ... < i_p$, $(0 \leq p \leq n)$, and for each such sequence $I \in \Sigma$ let e_I be the corresponding element $e_I = e_{i_1}...e_{i_p}$ of the basis of $\mathrm{Cl}(q)^p \subset \mathrm{Cl}(q)$. For $p = 0$ we have the empty set, and we take $e_\emptyset = 1 \in \mathrm{Cl}(q)^0$. Now every element a of $\mathrm{Cl}(q)$ can be uniquely written as

$$a = \sum_{I \in \Sigma} a_I e_I. \tag{127}$$

The coefficients a_I depend on the choice of the orthogonal basis e_i - except of the coefficient $a_\emptyset$, the scalar part of a. We denote it $\mathcal{T}(a)$ and call *the trace.* Thus we have defined a linear functional on the Clifford algebra $\mathrm{Cl}(q)$, with values in the basic field.

Definition 1.44. *We denote by $\mathcal{T}$ the linear functional on $\mathrm{Cl}(q)$ assigning to each element $a \in \mathrm{Cl}(q)$ its scalar part $a_\emptyset \in \mathrm{Cl}(q)^0$ in the direct sum decomposition (126).*

In the Proposition below we denote by $a \mapsto a^\tau$ the main anti-involution of $\mathrm{Cl}(q)$ (cf. Sec. 1.2.2). It is characterized by the following properties: $1^\tau = 1$, $x^\tau = x$ for $x \in M$, $(e_{i_1}...e_{i_p})^\tau = e_{i_p}...e_{i_1}$.

Proposition 1.45. *The functional $\mathcal{T}$ has the following properties:*

(i) $\mathcal{T}(1) = 1$,

(ii) $\mathcal{T}(a^\tau) = \mathcal{T}(a)$, $\forall a \in \mathrm{Cl}(q)$,

(iii) $\mathcal{T}(ab) = \mathcal{T}(ba)$, $\forall a, b \in \mathrm{Cl}(q)$,

(iv) $\mathcal{F}(a,b) \stackrel{df}{=} \mathcal{T}(a^\tau b)$ *is a symmetric, bilinear form on* $\mathrm{Cl}(q)$, *that is non-degenerate if F is a non-degenerate form on M. We have* $\mathcal{T}(a) = \mathcal{F}(1,a) = \mathcal{F}(a,1)$, $\forall a \in \mathrm{Cl}(q)$.

(v) $\mathcal{F}(ab,c) = \mathcal{F}(b, a^\tau c) = \mathcal{F}(a, cb^\tau)$, $\forall a,b,c \in C(Q)$.

Proof. (i) and (ii) follow immediately from the definition. In order to prove (iii) we notice that if e_i is an orthogonal basis in M, e_I, $I = \{i_1 < ... < i_p\}$ is the corresponding basis in $\mathrm{Cl}(q)$, and $a = \sum_I a_I e_I$, $b = \sum b_J e_J$. We notice that e_i and e_j anticommute for $i \neq j$ and that $e_i e_i = F(e_i, e_i)$ are scalars. Therefore $e_I e_J$ is proportional to e_K where K contains the indices that are in I but not in J, or in J but not in I (the symmetric difference of the sets I and J). Therefore $\mathcal{T}(ab) = \mathcal{T}(\sum_I a_I e_I b_J e_J) = \sum_I a_I b_I \mathcal{T}(e_I e_I) = \mathcal{T}(ba)$. That $\mathcal{F}$ is a symmetric bilinear form follows immediately from (ii) and (iii). F is non-degenerate if and only if all $F(e_i, e_i)$ are non zero, and it is immediate that this happens if and only if all $\mathcal{F}(e_I, e_I)$ are non-zero. The remaining statements follow easily from the definitions and the properties proven above. $\square$

1.5 The Clifford group

We assume that M is a finite dimensional vector space over a field with characteristic $\neq 2$, and that q is a nondegenerate quadratic form on M. We denote by $O(q)$ the group of invertible mappings $g : M \to M$, $x \mapsto gx$, that leave q invariant: $q(gx) = q(x)$, $\forall x \in M$. We denote by $SO(q)$ the subgroup of $O(q)$ of those $g \in O(q)$ that have determinant 1.

Let $\mathrm{Cl}(q)$ be the Clifford algebra of q and let $F(x, y)$ be the symmetric bilinear form such that $q(x) = F(x, x)$. In particular we have

$$xy + yx = 2F(x, y)\, \forall x, y \in M. \tag{128}$$

The invertible elements $u \in \mathrm{Cl}(q)$ form a group. In particular every vector $x \in M$ such that $q(x) \neq 0$ is invertible, and every product $x_1...x_k$ of such vectors is invertible. Indeed, if $q(x) \neq 0$, then $x^2 = q(x) \neq 0$, therefore $x^{-1} = x/q(x)$. And $x_1...x_k$ is invertible since the product of invertible elements is invertible.

Definition 1.46. *We define* the Clifford group $\Gamma = \Gamma(q)$ *to be the group of all invertible elements* $u \in \mathrm{Cl}(q)$ *which have the property that* uyu^{-1} *is in* M *whenever* y *is in* M. *We define* $\Gamma(q)^{\pm}$ *as the intersection of* $\Gamma(q)$ *and* $\mathrm{Cl}(q)_{\pm}$.

In general an arbitrary invertible element of the algebra will not have such a property. But, for instance, if $x \in M$ is invertible, then, with $y \in M$ we have

$$xyx^{-1} = (xy + yx - yx)x^{-1} = 2F(x, y)x^{-1} - y = \frac{2F(x, y)}{F(x, x)}x - y \in M. \tag{129}$$

Definition 1.47. *Let* $x \in M$ *be a vector with* $q(x) \neq 0$. *We define the reflection* τ_x *with respect to the hyperplane orthogonal to* x *as the linear transformation* $\tau_x : M \to M$ *defined by the formula*

$$\tau_x(y) = y - \frac{2F(x, y)}{F(x, x)}x. \tag{130}$$

It can be easily verified that hyperplane reflections are orthogonal transformations, that is $F(\tau_x(y_1), \tau_x(y_2)) = F(y_1, y_2)$.

Comparing now Eqs. (129) and (130) we see that

$$xyx^{-1} = -\tau_x(y). \tag{131}$$

The following theorem about orthogonal transformations and reflections is well known under the name of Cartan-Dieudonné theorem (see e.g. [13, p. 18].

Theorem 1.48 (Cartan-Dieudonné). *Let M be a vector space of finite dimension n over the field of characteristic $\neq 2$, and let q be a nondegenerate quadratic form on M. Then every orthogonal transformation $\sigma \in O(q)$ is a product of at most n hyperplane reflections.*

Notice that if y is in the hyperplane orthogonal to x, i.e. if $F(x, y) = 0$, then $\tau_x(y) = y$. On the other hand, if y is proportional to x, say $y = \alpha x$ for some scalar α, then

$$\tau_x(y) = \alpha x - \frac{2F(x, \alpha x)}{F(x, x)} x = \alpha x - 2\alpha x = -\alpha x = -y.$$

Let now u be in Γ. We define the mapping $\chi(u) : M \to M$ by the formula

$$\chi(u)(x) = uxu^{-1}. \tag{132}$$

Clearly $\chi(u)$ is a linear invertible transformation of the vector space M. In fact $\chi(u)$ is an orthogonal transformation, that is $\chi(u)$ is in $O(q)$. Indeed, we have

$$q(\chi(u)(x)) = (\chi(u)(x))^2 = uxu^{-1}uxu^{-1} = ux^2u^{-1} = q(x)uu^{-1} = q(x).$$

It follows easily from the very definition that $\chi : \Gamma(q) \to O(q)$ is a group homomorphism, and that $\chi(u) = \chi(u')$ if and only if $u' = \alpha u$, where α is an invertible element of center $Zq)$ of $\mathrm{Cl}(q)$.

The following theorem taken from Bourbaki [5][p. 151] collects important properties of the homomorphism χ.

Theorem 1.49. *Let n be the dimension of M. If n is even, then $\chi(\Gamma) = O(q)$ and $\chi(\Gamma^+) = SO(q)$. If n is odd then $\chi(\Gamma) = \chi(\Gamma^+) = SO(q)$.*

Every element $u \in \Gamma$ is of the form $u = \alpha u'$, where α is an invertible element of the center $Z(q)$ and $u' \in \Gamma$ is either even or odd. ■

The proposition below gives us the most general form of elements of Γ^+.

Proposition 1.50. *Every element $u \in \Gamma^+$ is a product of an even number of vectors $x_i \in M$, with $q(x_i) \neq 0$, $(i = 1, ..., 2k)$*

$$u = x_1....x_{2k}.$$

Proof. From the Theorem 1.49 we know that $\chi(u)$ is in $SO(q)$. From the Cartan-Dieudonné theorem 1.48 we know that $\chi(u)$ is a product of a certain number of reflections. Each reflection has determinant -1, while $\chi(u)$ has determinant $+1$, therefore $chi(u)$ is a product of an even number of reflections.

Let $x_1, ..., x_{2k}$ be the vectors defining these reflections, and let $u' = x_1....x_{2k}$. Then $\chi(u') = \chi(u)$. That is because every x_i implements the reflection with the minus sign (cf. Eq. (131), and there is an even number of such reflections. It follows that $u' = \alpha u$ where α is an invertible element of the center $Z(q)$. Replacing x_1 with$(1/\alpha)x_1$ we get the desired form. □

Remark 1.51. *Suppose now u is a general element of the Clifford group Γ. From the second part of Theorem 1.49 we know that $u = \alpha u'$, where α is an invertible element of the center. If u' is even, then, from the last proposition, we know that u' is a product of an even number of invertible vectors $x_i \in M$. Suppose now that u' is odd. Let x be any invertible vector in M. Then $u'x$ is even, it belongs to Γ^+, and therefore $u'x = x_1...x_{2k}$. Setting $x_{2k+1} = (1/q(x))x$ we obtain $u' = x_1....x_{2k+1}$. This way we obtained a general form of an arbitrary element of the Clifford group Γ.*

1.5.1 The spinor norm

With the assumptions as above for every element $u \in \Gamma(q)$ we define the spinor norm $N(u)$ by the formula

$$N(u) = \tau(u)u, \tag{133}$$

where τ is the main involution of the Clifford algebra $\mathrm{Cl}(q)$.

We notice that if u is in Γ, then also $\tau(u)$ is in Γ. Indeed, if, for $y \in M$, we have $uyu^{-1} = y'$, and since $\tau(u)^{-1} = \tau(u^{-1})$, and $\tau(y) = y$, $\tau(y') = y'$, we obtain $\tau(u)^{-1}y\tau(u) = y'$, therefore $\tau(u)^{-1}$ is in Γ. and, since Γ is a group, also $\tau(u)$ is in Γ.

It follows that N is a mapping $N : \Gamma \to \Gamma$. In fact it maps Γ into its center: for all $u \in \Gamma$ we have that $N(u)$ is in the center $Z(q)$ of $\mathrm{Cl}(q)$. The proof goes as follows: from $uyu^{-1} = y'$ we get $uy = y'u$. Applying τ to both sides we get $y\tau(u) = \tau(u)y'$. Multiplying by u from the right we obtain

$$y\tau(u)u = \tau(u)y'u = \tau(u)uu^{-1}y'u = \tau(u)uy.$$

Therefore $N(u)$ commutes with all $y \in M \subset \mathrm{Cl}(q)$, and thus it commutes with all elements of the algebra $\mathrm{Cl}(q)$.

From the definition it follows immediately that if $u \in \Gamma$ and $\alpha \neq 0$ is a scalar, then

$$N(\alpha u) = \alpha^2 N(u). \tag{134}$$

The next important property of the spinor norm N is that it is a group homomorphism, namely that $N(st) = N(s)N(t)$ for all $s, t \in \Gamma$. Indeed, using the fact that $N(s)$ is in the center of the algebra, we have

$$\tau(st)st = \tau(t)\tau(s)st = \tau(t)N(s)t = N(s)\tau(t)t = N(s)N(t).$$

All the above properties of N can be also deduced directly from a general form of elements of Γ discussed in Remark 1.51.

Definition 1.52. *The following groups are called spin groups:*

$$\begin{aligned}
\mathrm{Pin}(q) &:= \{s \in \Gamma(q)^{+} \cup \Gamma(q)^{-} : N(s) = \pm 1\}\\
\mathrm{Spin}(q) &:= \{s \in \Gamma(q)^{+} : N(s) = \pm 1\}\\
\mathrm{Spin}^{+}(q) &:= \{s \in \Gamma(q)^{+} : N(s) = +1\}.
\end{aligned} \tag{135}$$

1.5.2 Example $|Spin(3) \simeq \mathrm{SU}(2)$

We consider the Clifford algebra Cl_3 in the matrix realization as in Sec. 1.3.2.1.6. We have $\mathrm{Cl}_3 \simeq \mathrm{Mat}(2,\mathbb{C})$ with the main automorphism α realized as

$$\alpha : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix} \tag{136}$$

and the main anti-automorphism as

$$\tau(a) = a^{*},\ u \in \mathrm{Mat}(2,\mathbb{C}), \tag{137}$$

as in Eqs. (75), (76).

The even subalgebra consists of 2×2 complex matrices $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $\alpha(u) = u$, that is

$$\bar{a} = d,\ \bar{b} = -c. \tag{138}$$

The three vectors of the orthonoarmal basis are represented by Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. Since we have $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$, and $\sigma_3\sigma_1 = i\sigma_2$ the even subalgebra is generated by the matrices $i\sigma - i$, $(i = 1,2,3)$ with squares -1 - it is isomorphic to the algebra of quaternions.

The condition for the Spin group is $N(u) = \pm 1$, where $N(u) = \tau(u)u$. In our case $N(u) = u^{*}u$, and $u^{*}u$ is a Hermitian matrix with nonnegative eigenvalues. Therefore in our case $Spin(3) = \mathrm{Spin}_{+}(3)$ and $N(u) = 1$ means that $u^{*}u = 1$ i.e. u is a unitary matrix. Explicitely, with $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we have:

(i) $\bar{a}a + \bar{c}c = 1$,

(ii) $\bar{a}b + \bar{c}d = 0$,

(iii) $\bar{b}a + \bar{d}c = 0$,

(iv) $\bar{b}b + \bar{d}d = 0$.

The second and the third equations are not independent, one being the complex conjugate of the other. Substituting now the conditions in Eqs. 138 we find that (ii) and (iii) are satisfied automatically, while (i) and (iv) reduce to just one condition: $ad - bc = 1$, or $\det(u) = 1$ Thus u is in Spin(3) if and only if u is unitary of determinant 1. The group of all such matrices, the special unitary group in two complex dimensions, is denoted SU(2).

1.5.3 Example: Spin and Pin for signatures $(3,1)$ and $(1,3)$

There are two conventions for defining the standard quadratic form for the Minkowski space-time of special relativity. It is also called the Minkowski space *metric* because the associated bilinear form defines the flat Riemannian metric of the Minkowski spacetime. The $(3,1)$ signature, where space enters with positive sign Euclidean metric, is sometimes called West Coast convention, perhaps because Feynman was using it at Caltech. The opposite convention, when the plus sign is reserved for the time (or energy) component is then referred to as East Coast convention, because Schwinger was using it while at Harvard and MIT[9].

Minkowski space metric has been introduced be Einstein in his formulation of special relativity theory because in this metric the points on the hypersurface $q(x) = 0$ define the light cone with apex at the origin of coordinates. From this point of view it does not matter which convention is being used. In Secs. 1.3.2.1.9 and 1.3.2.1.10, when discussing Clifford algebras $\mathrm{Cl}_{3,1}$ and $\mathrm{Cl}_{1,3}$ we used coordinates (x_1, x_2, x_3, x_4) for the $(3,1)$ metric and coordinates (x_0, x_1, x_2, x_3) for the $(1,3)$ metric. Here, we want to compare the two cases, therefore we will use coordinates (x_0, x_1, x_2, x_3) for both signatures. The order of coordinates and their naming depends on convention. [10]

We introduce two bilinear forms, η and $\hat{\eta}$ defined by the matrices

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \hat{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{139}$$

[9] At the time of writing these notes a discussion of this subject is available on Peter Woit's blog entry "The West Coast Metric is the Wrong One" `https://www.math.columbia.edu/~woit/wordpress/?p=7773&cpage=1`

[10] The notation $\mathrm{Cl}_{p,q}$ in the Clifford algebra means that there are p pluses and q minuses in the quadratic form, and it has nothing to do with order of coordinates, which can be arbitrary.

or, in short

$$\eta = \mathrm{diag}(-1,1,1,1),\ \hat{\eta} = \mathrm{diag}(1,-1,-1,-1) = -\eta. \tag{140}$$

The Lorentz group O(3,1) is the same as O(1,3) - it consists of 4×4 real matrices $L = (L^\alpha{}_\beta)$ leaving the bilinear form η (or $\hat{\eta}$) invariant: $L^T\eta L = \eta$, or

$$L_\beta{}^\alpha \eta_{\alpha\gamma} L^\gamma{}_\delta = \eta_{\beta\delta}, \tag{141}$$

where L^T is the transpose of L: $L_\beta{}^\alpha = (L^T)_\alpha{}^\beta$. The special Lorentz groups SO(3,1) and SO(1,3), consisting of Lorentz matrices of determinant one are also the same. So are the time orientation preserving subgroups $\mathrm{SO}^\uparrow(3,1) = \mathrm{SO}^\uparrow(1,3)$ consisting of special Lorentz matrices that have $L^0{}_0 > 0$. But we know that the Clifford algebras for the two signatures are not isomorphic. We have $\mathrm{Cl}_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R})$ but $\mathrm{Cl}_{1,3} \simeq \mathrm{Mat}(2,\mathbb{H})$. Thus the question arises whether this difference of Clifford algebras may have some physical implications? The interested reader may wish to consult Ref. [2]

1.5.3.1 The group $\mathrm{Spin}^+(1,3) \simeq \mathrm{SL}(2,\mathbb{C})$ We start with identifying explicitely the spin group $\mathrm{Spin}^+(1,3)$ using the definition given in Eq. 135, and using the Weyl matrix representation of $\mathrm{Cl}_{1,3}$ - cf. Eq. 99. Here we recall it in a block matrix form using Pauli matrices - see Eq. (74):

$$\Gamma_0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix},\ \Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},\ (i = 1,2,3).$$

We can write it in one formula if we use the main automorphism α of the Clifford algebra Cl_3 which changes the sign of the three Pauli matrices:

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \alpha(\sigma_i) & 0 \end{pmatrix},\ (i = 0,..,3). \tag{142}$$

Thus every vector $x \in M \subset \mathrm{Cl}_{1,3}$ can be represented as a block matrix

$$x = \begin{pmatrix} 0 & X \\ \alpha(X) & 0 \end{pmatrix}, \tag{143}$$

where $X = \sum_{i=0}^3 x^i\sigma_i$ is a Hermitian 2×2 matrix.

In order to identify the group $\mathrm{Spin}^+(1,3)$ we will need to calculate the spin norm, and to calculate the spin norm we will need the explicit form of the main anti-automorphism τ. The explicit form of τ depends on the representation, and for the Weyl representation finding τ is rather easy. The

Hermitian conjugate is an involutive (i.e. its square is the identity) anti-automorphism of the full matrix algebra, but it does not suit our purpose because the matrices Γ_i for $i = 1, 2, 3$ are anti-Hermitian, while Γ_0 is hermitian. But adding the conjugation by Γ_0 does the job. Therefore, for the Weyl representation, we have

$$\tau(u) = \Gamma_0 u^* \Gamma_0^{-1},\ u \in \mathrm{Cl}_{1,3}. \tag{144}$$

We then need the explicit form of the even subalgebra $\mathrm{Cl}^+_{1,3}$. It is generated by the identity matrix, by the products $\Gamma_i\Gamma_j$, $(i < j)$, and by the matrix $\omega = \Gamma_0\Gamma_1\Gamma_2\Gamma_3$. We know from Eq. (143) that Γ_i are anti-diagonal block matrices of the form

$$\begin{pmatrix} 0 & X \\ \alpha(X) & 0 \end{pmatrix}, \tag{145}$$

where X is a Hermitian 2×2 matrix and $\alpha(X)$ is the same as defined by Eq. (75) in the discussion of the Clifford algebra Cl_3. Products of two such matrices will have the form

$$\begin{pmatrix} X\alpha(Y) & 0 \\ 0 & \alpha(X)Y \end{pmatrix}. \tag{146}$$

But, since α is an automorphism with $\alpha^2 = \mathrm{id}$, we can write these products as

$$\begin{pmatrix} A & 0 \\ 0 & \alpha(A) \end{pmatrix}, \tag{147}$$

where A is a complex 2×2 matrix. The matrix representing the identity $\mathbf{1}$ and the matrix representing ω will also have this form. Matrices of this form build a 4-dimensional real vector space, and the even algebra $\mathrm{Cl}^+_{1,3}$ is also $8-$ *dimensional*. Therefore the even subalgebra $\mathrm{Cl}^+_{1,3}$ is represented by matrices of the form (147), where $A \in \mathrm{Mat}(2, \mathbb{C})$.

In order to identify the *Spin*$^+$ group we have to look now at the condition $N(u) = 1$ for $u \in \mathrm{Cl}^+_{1,3}$. Setting

$$u = \begin{pmatrix} A & 0 \\ 0 & \alpha(A) \end{pmatrix}, \tag{148}$$

and using Eq. (144) we obtain

$$\tau(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & \alpha(A)^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha(A)^* & 0 \\ 0 & A^* \end{pmatrix}, \tag{149}$$

Therefore for $N(u) = \tau(u)u$ we obtain:

$$N(u) = \begin{pmatrix} \alpha(A)^* & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \alpha(A) \end{pmatrix} = \begin{pmatrix} \alpha(A)^*A & 0 \\ 0 & A^*\alpha(A) \end{pmatrix}. \tag{150}$$

Using now Eqs. (76)-(79) we obtain

$$N(u) = \begin{pmatrix} \det(A)\mathbf{1} & 0 \\ 0 & \det(A^*)\mathbf{1} \end{pmatrix}. \tag{151}$$

It follows that for the matrices from the even subalgebra the condition $N(u) = 1$ is equivalent to $det(A) = 1$ - the condition that characterizes the matrices from the group $\mathrm{SL}(2, \mathbb{C})$.

In order to identify the group Spin^+ we still need to implement the condition defining the Clifford group, that is the condition $uxu^{-1} \in M$ for all $x \in M$. However, in our particular case at hand, we will see that this condition is satisfied automatically if the condition $\det(A) = \pm 1$ is satisfied.

With x as in Eq. (143) and u as in Eq. (148) we obtain

$$uxu^{-1} = \begin{pmatrix} 0 & AX\alpha(A)^{-1} \\ \alpha(AX)A^{-1} & 0 \end{pmatrix}. \tag{152}$$

Now, $\det(A) = \pm 1$ is the same as $AA^\nu = A\alpha(A^*) = \pm 1$ i.e. $\alpha(A)^{-1} = \pm A^*$. Therefore

$$uxu^{-1} = \begin{pmatrix} 0 & X' \\ \alpha(X') & 0 \end{pmatrix}, \tag{153}$$

where $X' = \pm AXA^*$. It is clear that X' is Hermitian if X is Hermitian. Therefore the condition $uxu^{-1} \in M$ is indeed satisfied.

The same reasoning as above applies to the group $\mathrm{Spin}(1, 3)$. We conclude that $\mathrm{Spin}(1, 3)$ can be identified with the group of all 2×2 complex matrices of determinant ± 1.

If $\det(A) = -1$, then the orthogonal transformation in M is implemented as $X \mapsto X' = -AXA^*$. Since product of two transformations with determinant -1 is a transformation with determinant 1, it follows that every transformation characterized by $\det(A) = -1$ is a product of one particular transformation with $\det(A) = 1$, and some element of $\mathrm{SL}(2, \mathbb{C})$ that implements a special ortochronous Lorentz transformation. We can chose σ_3 as a particular matrix with determinant -1. In that case, if $X = \sum_{i=0}^{3} x^i \sigma_i$, then $X' = -\sigma_3 X \sigma_3^*$ has reversed coordinates x^0 and x^3. Therefore transformations from the group $\mathrm{Spin}_{1,3}$ implement orthogonal transformations from the group $\mathrm{SO}(1, 3)$, including time inversions, but always associated with inversions of some space axes, so that the determinant of the Lorentz transformation is always 1.

Even better it is to select the volume element $\omega = \Gamma_0\Gamma_1\Gamma_2\Gamma_3$, which i even and has $N(\omega) = -1$. It anticommutes with all four Γ-s therefore it implements the PT transformation that reverses the signs of all three space coordinates and reverses the direction of time: $X \mapsto X' = -X$.

1.5.3.2 The group $\mathrm{Spin}(3,1) \simeq \mathrm{Spin}(1,3)$ While discussing the Clifford algebra $\mathrm{Cl}_{3,1}$ in Sec 1.3.2.1.9we were using the Majorana (real) representation. Here it will more convenient to use a very simple modification of the Weyl antidiagonal representation. Namely we set $\tilde{\Gamma}_j = i\Gamma_j$, $(i = 0, ..., 3)$ and the matrices $\tilde{\Gamma}$ evidently represent the generators of $\mathrm{Cl}_{3,1}$ - they anticommute and $\tilde{\Gamma}_i^2 = -\Gamma_i^2$. For the identification of $\mathrm{Spin}_{3,1}$ we will need the main anti-automorphism of $\mathrm{Cl}_{3,1}$ in this representation - will denote it as $\tilde{\tau}$. While matrices Γ_i are all Hermitian, the matrices $\tilde{\Gamma}_i$ are all anti-Hermitian. We have $\tau(\tilde{\Gamma}_i) = -\tilde{\Gamma}_i$, but we need $\tilde{\tau}(\tilde{\Gamma}_i) = \tilde{\Gamma}_i$. We need a matrix that anticommutes with all anti-diagonal matrices. Such a matrix exists, we can take

$$\tilde{C} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = -i\omega = -i\tilde{\omega}. \tag{154}$$

Therefore

$$\tilde{\tau}(u) = \tilde{C}\tau(u)\tilde{C}^{-1}. \tag{155}$$

The products of two $\tilde{\Gamma}$ matrices differ only by a sign form the products of the corresponding Γ matrices. Therefore the even subalgebras ${\mathrm{Cl}_{1,3}}^+$ and ${\mathrm{Cl}_{3,1}}^+$ are the same. Moreover, the embedding of M into the two Clifford algebras are simply related $\tilde{M} = iM$, so that the conditions $uMu^{-1} = M$ and $u\tilde{M}u^{-1} = \tilde{M}$ are the same. Finally, we need to calculate the spinor norm $\tilde{N}(u)$ when applied to the elements of ${\mathrm{Cl}_{3,1}}^+$. But $\tilde{C}$ commutes with all block diagonal matrices, therefore $\tilde{N}(u) = N(u)$ for $u \in \mathrm{Cl}_{3,1}^+$. We conclude that the groups $\mathrm{Spin}_{3,1}$ and $\mathrm{Spin}_{1,3}$ are isomorphic. In fact, in our realization as groups of matrices, they are identical.

1.5.3.3 The groups $\mathrm{Pin}(1,3)$ **and** $\mathrm{Pin}(3,1)$ **are different** The elements Γ_0 and $\tilde{\Gamma}_0 = i\Gamma_0$ implement the same $\mathrm{O}(1,3) = \mathrm{O}(3,1)$ transformation - space inversion P. They belong to the groups $\mathrm{Pin}(1,3)$ and $\mathrm{Pin}(3,1)$ respectively. But $\Gamma_0^2 = 1$, while $\tilde{\Gamma}_0^2 = -1$. That is enough to see that the groups $\mathrm{Pin}(1,3)$ and $\mathrm{Pin}(3,1)$ are different, they are not isomorphic. Whether this fact may have some physical implication is not clear. Ref. [2] indicates that indeed that may be the case, while Ref. [11] proposes a different perspective.

2 Clifford algebra on multivectors

We assume, in this section, that M is vector space over reals or complex, not necessarily finite dimensional. Let $F(x, y)$ be a bilinear form (not necessarily symmetric) on M. We have seen in Proposition 1.26 that the mapping $\bar{\lambda}_F$ maps the Clifford algebra $C(q')$ of the quadratic form $q'(x) = q(x) + F(x, x)$

onto the Clifford algebra $\mathrm{Cl}(q)$. It is a vector space isomorphism, with the inverse mapping being $(\bar{\lambda}_F)^{-1} = \bar{\lambda}_{-F} : \mathrm{Cl}(q) \to C(q')$. Let us take the particular case of $q' = 0$ in which case the algebra $C(q')$ becomes identical to the exterior algebra $\Lambda(M)$. Elements of the exterior algebra are called multivectors and the multiplication of multivectors in the exterior algebra is traditionally denoted by the wedge symbol $x \wedge y$. But using the $\bar{\lambda}$ mapping we can also transport back to $\Lambda(M)$ the multiplication from the Clifford algebra $\mathrm{Cl}(q)$. We will now derive the corresponding formula. Let us take $x \in M \subset \Lambda(M)$ and $u \in \Lambda(M)$. Then $\bar{\lambda}_{-F}(x)$ and $\bar{\lambda}_{-F}(u)$ are in $\mathrm{Cl}(q)$. Now we multiply $\bar{\lambda}_{-F}(x)$ and $\bar{\lambda}_{-F}(u)$ in $c(q)$ and transport back their product $\bar{\lambda}{-}F(x)\bar{\lambda}_{-F}(u)$ to $\Lambda(M)$ using $\bar{\lambda}_F$. We obtain the multiplication rule of the Clifford algebra $\mathrm{Cl}(q)$ expressed in terms of multivectors:

$$xu = \bar{\lambda}_F(\bar{\lambda}_{-F}(x)\bar{\lambda}_{-F}(u)). \tag{156}$$

Notice that we identify the vectors of M with their images in $\mathrm{Cl}(q)$, therefore we can take $\bar{\lambda}_{-F}(x) = x$. We can then use Eq. (33):

$$xu = \bar{\lambda}_F(x\bar{\lambda}_{-F}(u)) = \bar{i}^F_x(\bar{\lambda}_F(\bar{\lambda}_{-F}(u))) + x \wedge \bar{\lambda}_F(\bar{\lambda}_{-F}(u)), \tag{157}$$

or

$$\boxed{xu = x \wedge u + \bar{i}^F_x(u).} \tag{158}$$

We recall the action of the antiderivation $\bar{i}^F_x$

(i) For all $x \in M$ we have

$$\boxed{\bar{i}^F_x(1) = 0,\ (1 \in \Lambda(M))}, \tag{159}$$

(ii) For all $x, y \in M \subset \Lambda(M)$, $w \in \Lambda(M)$, we have

$$\boxed{\bar{i}^F_x(y \wedge w) = F(x, y)w - y \wedge \bar{i}^F_x(w).} \tag{160}$$

The bilinear form F above is in general non-symmetric. It can be split as a sum of its symmetric part $F_s(x, y) = F_s(y, x)$ and antisymmetric part $F_a(x, y) = -F_a(y, x)$:

$$\begin{aligned} F(x, y) &= \frac{1}{2}(F(x, y) + F(y, x)) + \frac{1}{2}(F(x, y) - F(y, x)) \\ &= F_s(x, y) + F_a(x, y). \end{aligned} \tag{161}$$

From Eq. (160) we get

$$\begin{aligned} xy + yx &= 2F_s(x, y), \\ xy - yx &= 2(x \wedge y + F_a(x, y)). \end{aligned} \tag{162}$$

In particular Eq. (162) implies $x^2 = F_s(x,x)$. Therefore the multiplication defined in Eq. (158) determines the Clifford algebra $\mathrm{Cl}(q)$ with $q(x) = F_s(x,x)$, and q does not depend at all on the antisymmetric part F_a of F. And yet Eq. (158) defines different multiplications for different antisymmetric parts of F even if the symmetric parts are the same. However, it follows immediately from the universal property of the Clifford algebras that all these algebras corresponding to different antisymmetric parts of F are isomorphic one to another, as they are all Clifford algebras with the same q. Therefore it is somewhat surprising that in Ref. [1] Ablamowicz and Lounesto decided to take the trouble to verify this obvious property using a computer. They wrote

> "We explicitly demonstrate with a help of a computer that Clifford algebra C(B) of a bilinear form B with a non-trivial anti symmetric part A is isomorphic as an associative algebra to the Clifford algebra C(Q) of the quadratic form Q induced by the symmetric part of B." "

Moreover they attribute the formula (158 defining the Clifford multiplication for an arbitrary, possibly degenerate and not necessarily symmetric bilinear form on multivectors to Oziewicz [17] instead of referring to the classical old algebra book of Bourbaki, originally published by Hermann in 1959 [3].

2.1 The standard case of $\mathrm{Cl}(q)$

With the assumptions as above let F be the symmetric bilinear form with $q(x) = F(x,x)$. We then realize $\mathrm{Cl}(q)$ on $\Lambda(M)$ using the formulas (158)-(160). For $x, y \in M$ Eqs. (162) can then be written as

$$\begin{aligned} F(x,y) &= \frac{1}{2}(xy + yx), \\ x \wedge y &= \frac{1}{2}(xy - yx). \end{aligned} \tag{163}$$

Adding the two equations together we get

$$xy = x \wedge y + F(x,y), \tag{164}$$

which is another way of writing Eq. (158) for $u = y$. [5, Exercise 3, p. 154]

Proposition 2.1. *1. If $x, x_1, ..., x_n \in M$ then*

$$\bar{i}^F_x(x_1 \wedge ... \wedge x_n) = \sum_{i=1}^{n} (-1)^{i-1} F(x, x_i)\, x_1 \wedge ... x_{\hat{i}} ... \wedge x_n, , \tag{165}$$

where the symbol $x^{\hat{i}}$ means that x^i is omitted from the product.

2. If $x_i \in M$, $(i = 1, ..., n)$ are mutually orthogonal (i.e if $F(x_i, x_j) =$ for $i \neq j$) then

$$x_1 ... x_n = x_1 \wedge ... \wedge x_n. \tag{166}$$

Proof. Setting $y = x, w = x_1$ in Eq. (160) we get Eq. (165) for $n = 1$. Assuming that it holds for $n - 1$, setting $w = x_2 \wedge ... \wedge x_n$, from Eq. (160) we get

$$\begin{aligned} \bar{i}^F_x (x_1 \wedge x_2 \wedge ... \wedge x_n) &= F(x, x_1)\, x_2 \wedge ... \wedge x_n - x_1 \wedge \bar{i}^F_x (x_2 \wedge ... \wedge x_n) \\ &= F(x, x_1)\, x_2 \wedge ... \wedge x_n - x_1 \wedge \sum_{i=2}^{n} (-1)^{i-2} F(x, x_i)\, x_2 \wedge ... x_{\hat{i}} ... \wedge x_n \\ &= \sum_{i=1}^{n} (-1)^{i-1} F(x, x_i)\, x_1 \wedge ... x_{\hat{i}} ... \wedge x_n. \end{aligned} \tag{167}$$

To prove (166) we observe that it is true for $n = 2$ owing to Eq. (158). Assuming that it holds for $n - 1$, from Eq. (158) we have

$$x_1 (x_2 ... x_n) = x_1 (\wedge ... \wedge x_n) - \bar{i}_{x_1} (x_1 \wedge ... \wedge x_n) = x_1 \wedge ... \wedge x_n, \tag{168}$$

where we have used the orthogonality assumption and Eq. (165). □

Corollary 2.2. *For any finite number $x_1, ..., x_n$ in M we have:*

$$x_1 \wedge ... \wedge x_n = \frac{1}{n!} \sum_\sigma (-1)^\sigma\, x_{\sigma(1)} ... x_{\sigma(n)}, \tag{169}$$

where the sum on the right is over all $n!$ permutations σ of $(1, ..., n)$, and $(-1)^\sigma$ is the sign of the permutation: $(+1)$ for even, (-1) for odd permutation.

Proof. For $n = 2$ Eq. (169) is the same as Eq. (163). But we can prove it in a different way showing the idea of the proof for general n. We can choose an orthogonal system of vectors $e_1, ..., e_m$ such that x_1, x_2 are linear combinations of these vectors:

$$x_1 = x_1^i e_i,\ x_2 = x_2^j e_j,$$

where Einstein convention is used for the sum over the repeated indices, here i and j. Thus for the right-hand side of Eq. (168) we have

$$RHS = \frac{1}{2} \sum_\sigma (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} = \frac{1}{2}(x_1 x_2 - x_2 x_1) = \frac{1}{2} x_1^i x_2^j (e_i e_j - e_j e_i).$$

It is clear that the sum over i, j can be reduced to $i \neq j$. But then, according to Eq. (169) $e_i e_j = e_i \wedge e_j$, thus

$$RHS = \frac{1}{2} x_1^i x_2^j (e_i \wedge e_j - e_j \wedge e_i) = \frac{1}{2} x_1^i x_2^j (2\, e_i \wedge e_j) = x_1 \wedge x_2.$$

Exactly the same method works for general n

$$\begin{aligned} RHS &= \frac{1}{n!} \sum_\sigma (-1)^\sigma \, x_{\sigma(1)} ... x_{\sigma(n)} = \frac{1}{n!} x_1^{i_1} ... x_n^{i_n} \sum_\sigma (-1)^\sigma e_{i_{\sigma(1)}} ... e_{i_{\sigma(n)}} \\ &= \frac{1}{n!} x_1^{i_1} ... x_n^{i_n} \sum_\sigma (-1)^\sigma e_{i_{\sigma(1)}} \wedge ... \wedge e_{i_{\sigma(n)}} = \frac{1}{n!} x_1^{i_1} ... x_n^{i_n} n!\, e_{i_1} \wedge ... \wedge e_{i_n} \\ &= x_1 \wedge ... \wedge x_n. \end{aligned} \tag{170}$$

□

Corollary 2.3. *The anti-automorphism τ is the same for the exterior algebra $\Lambda(M)$ and for the Clifford algebra $\mathrm{Cl}(q)$ defined on $\Lambda(M)$ as in Eq. (164).*

Proof. If $\{e_i\}$ is an orthogonal basis, then $e_{i_1} ... e_{i_p}$, $i_1 < ... < i_p$ form a basis in $\mathrm{Cl}(q)$. But then

$$e_{i_1} .... e_{i_p} = e_{i_1} \wedge ... \wedge e_{i_p}$$

form a basis for $\Lambda(M)$, and τ acts the same way on these homogeneous elements by reversing the order. □

In Proposition 1.45 we defined the bilinear form $\mathcal{F}$ on $\mathrm{Cl}(q)$ as

$$\mathcal{F}(ab) = (a^\tau b)_0, \tag{171}$$

where $(a^\tau b)_0$ is the scalar (grade zero) part of the product $a^\tau b$.

Proposition 2.4. *For $x_1, ..., x_p, y_1, ..., y_p$ in M we have*

$$\mathcal{F}(x_1 \wedge ... \wedge x_p, y_1 \wedge ... \wedge y_p) = \det(F(x_i, y_j)). \tag{172}$$

Proof. Both sides are multilinear and antisymmetric with respect to x_i and y_j. Therefore it is sufficient to verify the equality for ordered basis vectors e_i. Then for the left hand side we get a non-zero expression only for

$$\mathcal{F}(e_{i_1} \wedge ... \wedge e_{i_p}, e_{i_1} \wedge ... \wedge e_{i_p}) = q(e_{i_1}) ... q(e_{i_p}). \tag{173}$$

And exactly the same expression we get for the determinant of the diagonal matrix on the right hand side. □

2.2 Maxwell equations

In a flat space-time Maxwell equations in a relativistic form can be represented as follows (Cf. e.g. [21, p. 166]):

$$\partial_i \mathfrak{F}_{kl} + \partial_k \mathfrak{F}_{li} + \partial_l \mathfrak{F}_{ik} = 0, \tag{174}$$

$$\eta^{jk} \partial_j \mathfrak{F}_{ik} = s_i, \tag{175}$$

where η^{jk} is the bilinear form defining the Minkowski metric, $\mathfrak{F}_{ij} = -\mathfrak{F}_{ji}$ is the electromagnetic field tensor, s_i is the current co-vector (differential 1-form), $\partial_i \mathfrak{F}_{jk}$ stands for $\partial \mathfrak{F}_{jk}/\partial x^i$, and we use the Einstein summation convention. We will show that Eqs. (174), (175) can be written as one equation

$$\partial\!\!\!/ \mathfrak{F} = s, \tag{176}$$

where $\mathfrak{F}$ and s are Clifford algebra valued functions and $\partial\!\!\!/$ is the Dirac operator.

Let M be the Minkowski space with coordinates x^i. The vectors (strictly speaking "tangent vectors", but we are not going to dive into differential geometry here) and contravariant tensors have components with upper indices, for instance v^i, while covariant tensors have components with lower indices, for instance s_i. We assume the space of covectors is equipped with the flat Minkowski metric η^{ij}. We denote by e^i the orthonormal basis of covectors $e^i = dx^i$, and let $\mathrm{Cl}(\eta)$ be the corresponding Clifford algebra. Vectors e^i, together with the relations $e^i e^j + e^j e^i = 2\eta^{ij}$ generate $\mathrm{Cl}(\eta)$. We realize this Clifford algebra product on the space of differential forms $\Lambda(M^*)$, as we have realized it on multivectors $\Lambda(M)$ before. Electromagnetic field tensor is represented by a differential form $\mathfrak{F} \in \Lambda^2(M^*)$ of the second order:

$$\mathfrak{F} = \frac{1}{2} \mathfrak{F}_{kl} e^k \wedge e^l. \tag{177}$$

The electric current is represented by 1-form $s \in \Lambda^1(M^*)$:

$$s = s_l \, e^l. \tag{178}$$

Both $\mathfrak{F}_{ij}$ and s^i are assumed to be functions of coordinates $\mathfrak{F}_{ij} = \mathfrak{F}_{ij}(x^k)$, $s_i = s_i(x^k)$. The Dirac operator $\partial\!\!\!/$ acting on functions with values in $\mathrm{Cl}(\eta)$ is defined as

$$\partial\!\!\!/ = e^i \partial_i. \tag{179}$$

We now analyze Eq. (176) and show that it is equivalent to the pair of equations (174), (175). We have

$$\partial\!\!\!/ \mathfrak{F} = \frac{1}{2} \partial_i \mathfrak{F}_{kl} \, e^i e^k \wedge e^l, \tag{180}$$

and we will now calculate $e^ie^k \wedge e^l$ using Eqs. (158) and (165). We have

$$e^i\, e^k e^l = e^i \wedge e^k \wedge e^l + \bar{i}^{\eta}_{e_i}(e^k \wedge e^l), \tag{181}$$

and

$$\bar{i}^{\eta}_{e_i}(e^k \wedge e^l) = \eta^{ik} e^l - \eta^{il} e^k. \tag{182}$$

When contracted with the antisymmetric $\mathfrak{F}_{kl}$ the two terms will give the same contribution so that the factor 1/2 will disappear. Therefore:

$$\partial\!\!\!/ \mathfrak{F} == \frac{1}{2}\partial_i \mathfrak{F}_{kl}\, e^i \wedge e^k \wedge e^l + \eta^{ik}\partial_i \mathfrak{F}_{kl}\, e^l. \tag{183}$$

2.3 The Dirac operator

With the assumptions as in above, in Sec. 2.2, we will now analyze the Dirac operator acting on a general homogeneous element $\mathfrak{F}^p = \frac{1}{p!}\mathfrak{F}_{i_1 \ldots . i_p} e^{i_1} \wedge \ldots \wedge e^{i_p} \in \Lambda^p(M^*)$. where $\mathfrak{F}_{i_1 \ldots . i_p} = \mathfrak{F}_{i_1 \ldots . i_p}(x)$ are antisymmetric with respect to the indices $x_1, ..., i_p$ functions of the coordinates $x^1, ..., x^n$. In other words $\mathfrak{F}$ is a p-form on M and $\mathfrak{F}_{i_1 \ldots . i_p}(x)$ are its components, which are functions om M. The Dirac operator $\partial\!\!\!/$ acting on $\mathfrak{F}^p$ will have, as before, two parts:

$$\partial\!\!\!/ \mathfrak{F}^p = \overset{p+1}{\mathfrak{F}} + \overset{p-1}{\mathfrak{F}}, \tag{184}$$

where

$$\overset{p+1}{\mathfrak{F}} = \frac{1}{p!}\partial_i \mathfrak{F}_{i_1 \ldots i_p}\, e^i \wedge e^{i_1} \wedge \ldots \wedge e^{i_p} \tag{185}$$

is a $(p+1)$-form, and

$$\overset{p-1}{\mathfrak{F}} = \frac{1}{p!}\partial_i \mathfrak{F}_{i_i \ldots i_p}\, \bar{i}^{F}_{e^i}(e_{i_1} \wedge \ldots \wedge e^{i_p}) \tag{186}$$

is a $(p-1)$-form. If $p = n$ then $\overset{p+1}{\mathfrak{F}} = 0$ and if $p = 0$ then $\overset{p-1}{\mathfrak{F}} = 0$. Moreover, the first part, $\overset{p+1}{\mathfrak{F}}$ does not depend on the bilinear form F. In fact, it is known under the name "exterior derivative" and denoted as d. Thus we have

$$\overset{p+1}{\mathfrak{F}} = d\mathfrak{F}^p. \tag{187}$$

Since $d\mathfrak{F}^p$ is a $(p+1)$-form, it is expressed in terms of its components as

$$d\mathfrak{F}^p = \frac{1}{(p+1)!}(d\mathfrak{F}^p)_{i_1 \ldots i_{p+1}}\, e_{i_1} \wedge \ldots \wedge e_{i_{p+1}}). \tag{188}$$

In order to get the expression for the components $(d\overset{p}{\mathfrak{F}})_{i_1\ldots i_{p+1}}$ we need to antisymmetrize the right hand side in Eq. (185). We first change the names of the summation indices:

$$\overset{p+1}{\mathfrak{F}} = \frac{1}{p!}\partial_{i_1}\mathfrak{F}_{i_2\ldots i_{p+1}}\, e^{i_1}\wedge e^{i_2}\wedge \ldots \wedge e^{i_{p+1}}, \tag{189}$$

and then antisymmetrize by replacing $\partial_{i_1}\mathfrak{F}_{i_2\ldots i_{p+1}}$ with

$$\frac{1}{p+1}\sum_{k=1}^{p+1}(-1)^{k-1}\partial_{i_k}\mathfrak{F}_{i_1\ldots\hat{i_k}\ldots i_{p+1}}.$$

We obtain

$$\overset{p+1}{\mathfrak{F}} = \frac{1}{(p+1)!}\sum_{k=1}^{p+1}(-1)^{k-1}\partial_{i_k}\mathfrak{F}_{i_1\ldots\hat{i_k}\ldots i_{p+1}}\, e^{i_1}\wedge e^{i_2}\wedge \ldots \wedge e^{i_{p+1}}, \tag{190}$$

and therefore

$$(d\mathfrak{F})_{i_1\ldots, i_{p+1}} = \sum_{k=1}^{p+1}(-1)^{k-1}\partial_{i_k}\mathfrak{F}_{i_1\ldots\hat{i_k}\ldots i_{p+1}}. \tag{191}$$

Eq. (191) is the standard expression for the exterior derivative of a p-form. Notice that $d^2 = 0$ because the expression for the components of $d^2\mathfrak{F}$ will contain second derivatives of the components of $\mathfrak{F}$. The components of $d^2\mathfrak{F}$ should be all antisymmetric, but mixed derivatives are symmetric. Therefore $d^2\mathfrak{F}$ must be zero.

3 Deformations

Here we will expand the method used in the previous section to include more general *deformations* of Clifford algebras. We will start with presenting the facts discussed before from a somewhat more general perspective. We will start assuming that *M is a vector space over the field R of an arbitrary characteristic (thus including characteristic 2).*

3.1 The additive group of bilinear forms Bil(M)

We will deal with three important sets: the set of all bilinear forms Bil(M), the set of all alternate forms Alt(M), and the set of all quadratic forms Quad(M).[11] Each of these sets is, in fact, a vector space. But we will be

[11] I am following the notation used in Ref. [8].

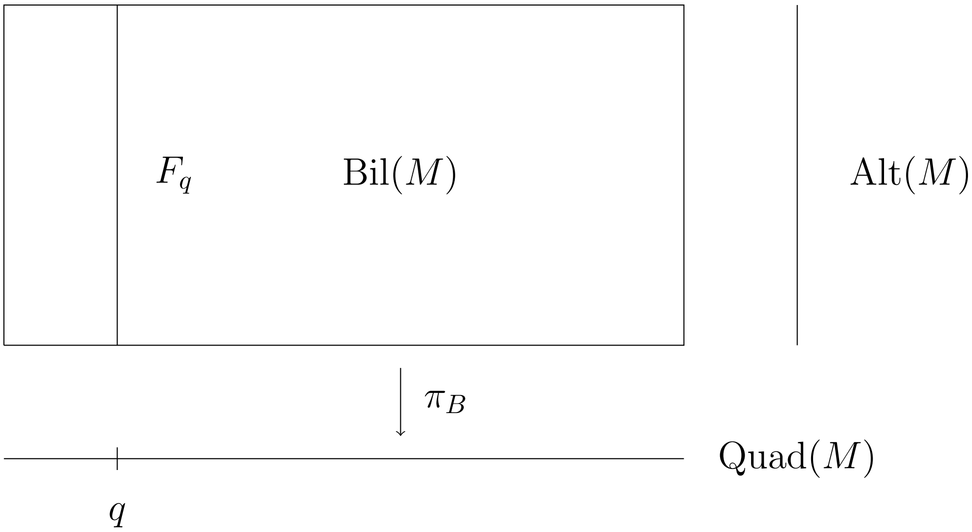

Figure 1: Principal bundle Bil(M) of bilinear forms over the base Quad(M) of quadratic forms, with structure group Alt(M) of alternate forms. The fiber F_q over q consists of all bilinear forms $F(x, y)$ such that $q(x) = F(x, x)$, i.e. $q = \pi_B(F)$.

mainly interested that these sets are *Abelian groups* with respect to the addition "+". We can associate with these sets the following diagram:

$$\mathrm{Alt}(M) \longrightarrow \mathrm{Bil}(M) \xrightarrow{\pi_B} \mathrm{Quad}(M), \tag{192}$$

meaning that Alt(M) is a subgroup of Bil(M) and that every quadratic form $q(x)$ can be obtained from some bilinear form $F(x, y)$ via $q(x) = F(x, x)$, with F and F' determining the same $q(x)$ if and only if $F'(x, y) - F(x, y) = A(x, y)$ where $A(x, y)$ is alternate, i.e. $A(x, x) = 0$ for all $x \in M$. This last property has been discussed in Remark 1.12. The mapping π_B associates with every bilinear form F the quadratic form $q(x) = F(x, x)$.

The sequence in Eq.(192) is called *exact*, which means that the map $\mathrm{Alt}(M) \to \mathrm{Bil}(M)$ is injective, and that Alt(M) is the kernel of the map π_B. What we have can be summarized by saying that we have a *principal bundle* - the group Bil(M) over the base Quad(M) - the homogeneous space $\mathrm{Quad}(M) = \mathrm{Bil}(M)/\mathrm{Alt}(M)$, as depicted in Fig. 1.

Remark 3.1. *Here and in the following we are using the language of fiber bundles in an informal way, without paying any attention whatsoever to topology, since topology is not needed in these general algebraic considerations. Topology will come back when we will specify the arbitrary field R to become real or complex numbers.*

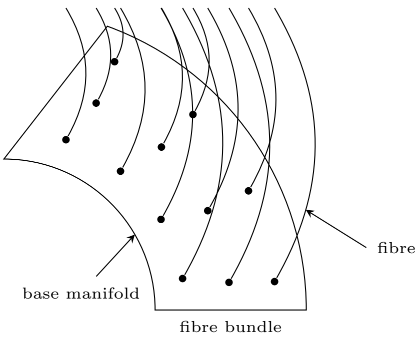

Figure 2: An artistic 3D drawing of a fiber bundle - inappropriate in our context

Sometimes fiber bundles are graphically represented three dimensionally as in Fig.2. That representation is inappropriate in our case, as it may suggest that each fiber has a distinguished point. But this is not the case in general. While in Remark 1.12 we have indeed constructed a bilinear form from a quadratic form, the construction there was dependent on the choice of a basis in M. Of course there is a distinguish point on each fiber when the field R admits division by 2 - in that case for each quadratic form there is a unique symmetric form in each fibre, namely $F(x,y) = \frac{1}{2}\Phi(x,y)$, where Φ is the bilinear form associated with q.

3.2 The bundle of Clifford algebras

For every quadratic form $q \in \mathrm{Quad}$ we have constructed (see Section 1.2) the Clifford algebra $\mathrm{Cl}(q) = T(M)/J(q)$. We denote by $C(M)$ the collection of all these Clifford algebras:

$$C(M) = \bigcup \{\mathrm{Cl}(q) : q \in \mathrm{Quad}(M)\}. \tag{193}$$

Then to give $C(M)$ the structure of a vector bundle over the base $\mathrm{Quad}(M)$, we need to provide it with local coordinates that enable us represent $\mathrm{Cl}(M)$ as a cartesian product of the base and of a vector space. In fact in our case we can provide not only local but also global coordinates. To this end let $\{e_i\}_{i\in I}$ be a basis in the vector space M, with a well ordered index set I. We the have the following important result (see [3, Theorem 1, p. 145]:

Theorem 3.2. *Assume that $\{e_i\}_{i\in I}$ is a basis in M, with a well ordered index set I. For every finite part H of I let us set $e_H = e_{i_1}\cdots e_{i_n}$ where $\{i_1,\cdots,i_n\}$ is the ordered sequence of all elements of H: $i_1 < \cdots < i_n$. Then the elements e_H, with H running through all finite subsets of I form a basis for* $\mathrm{Cl}(q)$

Proof. We follow the proof as given in Ref. [3, Theorem 1, p. 145], with only slight adaptations. The proof assumes that we already know that the result holds for the exterior algebra $\Lambda(M) = \mathrm{Cl}(0)$, within which context it is a standard property. Therefore $e_H = e_{i_1} \wedge \cdots \wedge e_{i_n}$ form a basis in $\mathrm{Cl}(0)$. Given now $q \in \mathrm{Quad}(M)$ we construct bilinear form $F(x, y)$ as in Remark 1.12, but this time for the form $-q$, and with reversed order, that is with $F(e_i, e_i) = -q(e_i)$,$F(e_i, e_j) = 0$ for $i < j$ and $F(e_i, e_j) = -\Phi(e_i, e_j)$ for $i > j$. In particular we have $q(x) + F(x, x) = 0$. The map $\bar{\lambda}_F$ of Proposition 1.24 provides now vector space isomorphism $\bar{\lambda}_F : \mathrm{Cl}(0) \to \mathrm{Cl}(q)$. We will now prove that

$$\bar{\lambda}_F(e_H) = \bar{\lambda}_F(e_{i_1} \wedge \cdots \wedge e_{i_n}) = e_{i_1} \cdots e_{i_n}, \tag{194}$$

where the multiplication on the right hand side is that in $\mathrm{Cl}(q)$. The proof of this last property is by induction. It is evident for $n = 1$, since $\bar{\lambda}_F(x) = x$ for every $x \in M$. Suppose it holds for all sequences $i_1 < \cdots < i_n$. We will show that then it also holds for sequences of length $n + 1$. We will use the fundamental property of $\bar{\lambda}_F$ in Eq. (33. Suppose H has $n + 1$ elements, and let j be its first element, with $H = \{j, i_1, \cdots i_n\}$, and $j < i_1 < \cdots < i_n$. Using Eq. (33 we have

$$\bar{\lambda}_F(e_H) = e_j \bar{\lambda}_F(e_{i_1} \wedge \cdots \wedge e_{i_n}) + i^F_{e_j}(\bar{\lambda}_F(e_{i_1} \wedge \cdots \wedge e_{i_n})). \tag{195}$$

By the induction hypothesis we have $\bar{\lambda}_F(e_{i_1} \wedge \cdots e_{i_n}) = e_{i_1} \cdots e_{i_n}$. Therefore

$$\bar{\lambda}_F(e_H) = e_j e_{i_1} \cdots e_{i_n} + \bar{i}^F_{e_j}(e_{i_1} \cdots e_{i_n}). \tag{196}$$

But now we use Eq. (26) and find the last term vanishes, because expanding it we will be getting terms with $F(e_j, e_{i_k})$ which vanish by the very construction of F. □

3.3 Automorphisms and deformations in the bundle of Clifford algebras

We have arrived at the following picture: We have action, let us denote it by $\tilde{\lambda}$, of the additive group of bilinear forms $\mathrm{Bil}(M)$ on the manifold $\mathrm{Quad}(M)$ the stability subgroup being $\mathrm{Alt}(M)$, the additive group of alternate forms:

$$\begin{array}{rcl} \tilde{\lambda} : \mathrm{Bil}(M) \times \mathrm{Quad}(M) & \to & \mathrm{Quad}(M) \\ \tilde{\lambda}(F, q) & = & q', \\ q'(x) & = & q(x) - F(x, x). \end{array} \tag{197}$$

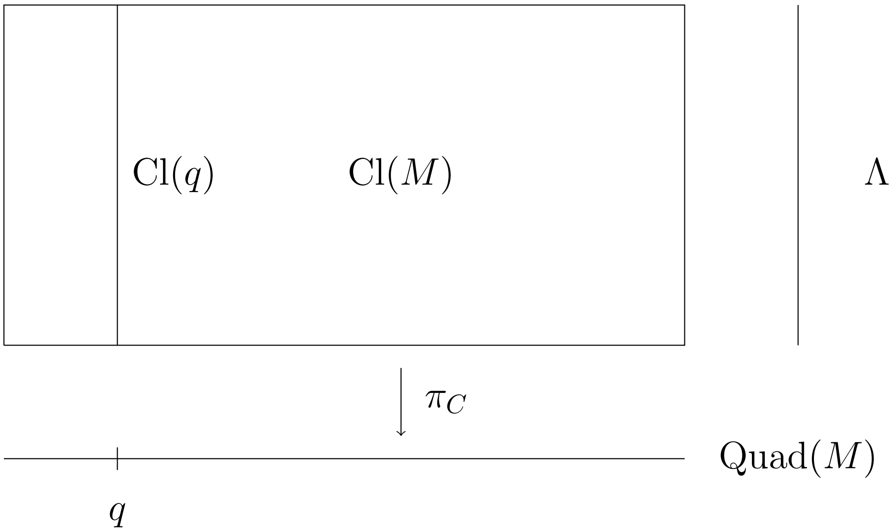

Figure 3: Vector bundle Cl(M) of Clifford algebras Cl(q) over the base Quad(M) of quadratic forms, with the exterior algebra $\Lambda(M)$ as a typical fibre. Global fibre coordinates are provided by selecting a basis in M, as shown in Theorem 3.2

[12] The group Bil(M) acts on the basis on the basis of the vector bundle Cl(M) whose fibers are Clifford algebras Cl(q). And we know that his action admits what is called a *lifting*, and we denote it with the letter $\bar{\lambda}$, to the bundle Cl(M) :

$$\bar{\lambda}(F,u) = \bar{\lambda}_{-F}(u),\ u \in \mathrm{Cl}(q), \tag{198}$$

where $\bar{\lambda}_F$ have been defined in Proposition 1.24. Now $\bar{\lambda}(F, C_q) = C(\tilde{\lambda}(q)) = C(q')$. Thus fibers are mapped onto fibers by linear isomorphisms - see Fig. 4. For $F \in \mathrm{Alt}(M)$ we have $q' = q$ and so each fiber Cl(q) is mapped linearly onto itself.

In each Clifford algebra Cl(q) we can now define a family of its deformations parameterized by bilinear forms $F \in \mathrm{Bil}(M)$. We do it the same way as we have introduced Clifford algebra structure in the exterior algebra. Given $F \in \mathrm{Bil}(M)$ we define new algebra product $\cdot_F$ in Cl(q) using the formula:

$$u \cdot_F w = \bar{\lambda}_F(\bar{\lambda}_{-F}(u)\bar{\lambda}_{-F}(w)),\ (u, w \in \mathrm{Cl}(q)). \tag{199}$$

The new product so defined is automatically associative.[13] The formula

[12]Here we have defined the action as a subtraction rather than as an addition because of the convention already taken in Proposition 1.24, where $\bar{\lambda}_F$ was defined as a mapping from $C(q')$ to Cl(q) rather than from Cl(q) to $C(q')$. Here we are exchanging in our notation q and q'

[13]It is quite general and almost evident.Let A be a set, B be an algebra, and $T : A \to B$

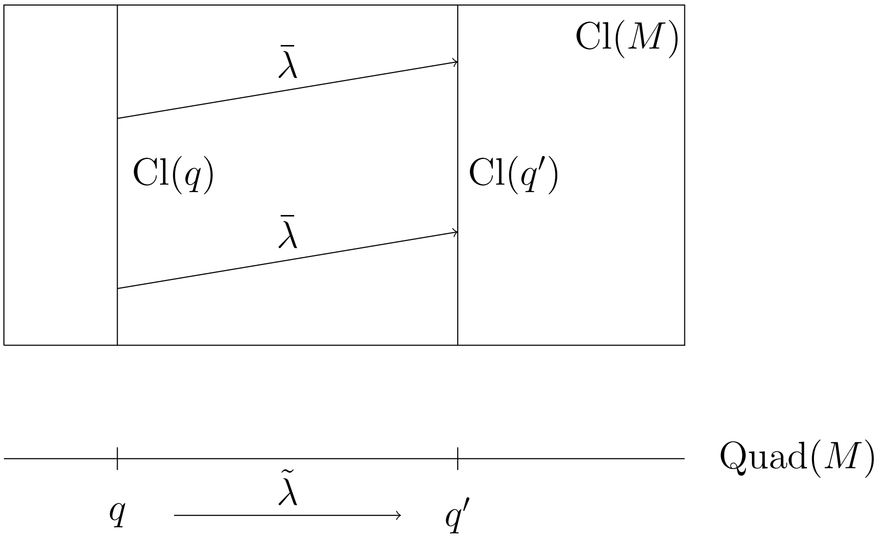

Figure 4: Every bilinear form $F \in \mathrm{Bil}(M)$ defines a automorphism of the vector bundle of Clifford algebras mapping linearly fibers onto fibers. Alternate forms in $\mathrm{Alt}(M) \subset \mathrm{Bil}(M)$ define vertical automorphisms - they do not move points on the base and map every fiber into itself. Such automorphisms are also called *gauge transformations*

defining explicitly the new multiplication in $\mathrm{Cl}(q)$ can be derived exactly the same way as we have derived the formula (158:

$$\boxed{x \cdot_F u = xu + \bar{i}^F_x(u).}, \tag{200}$$

where, for $x, y \in M,\ u, w \in \mathrm{Cl}(q)$ we have

$$\boxed{\bar{i}^F_x(yw) = F(x,y)w - y\bar{i}^F_x(w),} \tag{201}$$

and the multiplications on the right in (200) and on the left in (201) are in $\mathrm{Cl}(q)$.

In Ref. [10], in Section 4.7, *Deformations of Clifford algebras*, the formula completely equivalent to Eq. (158) is derived using rather advanced algebraic manipulations and associativity necessitates a complicated almost one page proof.

be a linear map. Let $\cdot$ be the product defined in A as $a \cdot b = T^{-1}(TaTb)$. Then

$$(a \cdot b) \cdot c = T^{-1}(\,T(T^{-1}(TaTb))Tc\,) = T^{-1}(\,TaTbTc\,)$$

, and associativity follows from the associativity of the product in B.

The formula given in Ref. [10] also involves a certain *exponential*. We will see how exponential enters in our case in a way analogous to our discussion of the mapping λ_F as an exponential.

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