Comments on Chapter 5 of G. I. Shipov's "A Theory of Physical Vacuum". Part I

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***Abstract*—The paper discusses mathematical problems and inconsistencies in Ch. 5 of the monograph "A Theory of Physical Vacuum" by G. I. Shipov. Particular attention is paid to sections 5.4 and 5.5 , where Cartan's formalism of moving frames and differential forms is improperly employed for the study of spacetime absolute parallelism geometry with torsion. Similar or identical problems in other publications of the same author are are pointed out. Mathematical inconsistencies found are listed, and the proper way of addressing the subject is indicated.**

***Index Terms*—torsion, absolute parallelism, Cartan's method, teleparallel geometry**

I. Introduction

The object of these comments is a part of Chapter 5 of the monograph by G. I. Shipov "A Theory of Physical vacuum" [1]. I will refer to this text as "The Book" and to the author as "Author". While concentrating on the book, once in a while, I will make comments about other works of the Author, those that contain parts that are almost identical to the relevant parts of the book. The problems found in the treatment of spin motion in a teleparallel geometry with torsion will be dealt with in the future, in Part II of this series.

Writing a critical review of someone's else work is a heavy responsibility. In pointing out mistakes and errors I did not want to make errors myself. For this reason I asked F. W. Hehl, with whom I coauthored another critical paper on torsion theories [2], whether he would be willing to help me with his advice in this task. His reply was kind, yet it contained the statement that he has lost enough time with Myron Evans - [2], he does not want now to lose time with Shipov. Then he asked me if I really think that it is worthwhile to look into this. I asked essentially the same question to my other colleague with whom I coauthored a monograph on Riemannian geometry [3]. He replied that, while he likes torsion, there are plenty of books with zillions of mistakes, so why do I want to spend time on this particular one? My answer was: "I am interested in the subject and I want to know the truth".

And this is the main reason for these notes in which I am pointing out those mistakes and errors in the book that caught my eye. Yes, there are plenty of them; some can be easily corrected, some seem to be serious; yet it would be a mistake to generalize and deduce from the plentitude of errors in one part, that everything else in that book must also be wrong. For instance: the second edition of the famous monograph by Kobayashi and Nomizu on differential geometry [4] contains a whole two pages of errata. In spite of this, this book still has errors. Yet this is a really good book. The companion volume to another excellent monograph "Analysis, Manifold and Physics" by Y. Choquet-Bruhat and C. DeWitt-Morette [5] has ten pages of errata to Part I; whole theorems, together with their proofs had to be replaced.

"*Errare humanum est. Stultum est in errore preservare*[1]." The whole point is to be able to correct those errors that can be corrected, to admit and to stop the propagation of those that can not be fixed, and to learn from them.

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[1] Attributed to the stoic Roman philosopher Lucius Annaeus Seneca. Translating into Russian: Человеку свойственно ошибаться, но глупо упорствовать в своих ошибках.

II. Notation and terminology

I will assume that the reader is familiar with the basic concepts of differential geometry, in particular *the theory of connections in vector bundles*. Such a knowledge is, in fact, assumed in the mathematical part of all publications about "*torsion fields*". An extensive overview of the relevant mathematical concepts can be found in a comprehensive review by Eguchi, Gilkey and Hanson [6]. If needed, additional information can be found in references [4], [7]–[15]. [2]

When comparing different sources we will often find that the authors may use different conventions for naming and labelling the same quantities, therefore care is necessary. The Book itself introduces its own naming and labelling conventions, which makes the task of analysing its content even harder. In order to facilitate the comparison of the content of the book with standard text, *I will use Shipov's notation and conventions*, and relate them to those that can be found in the literature dealing with same subject.

Once in a while I will quote the original formulas from the book. In these cases I will use double parenthesis, e.g. ((5.88)) for equation labels as in the original.

A. Affine connection

The main object of the study is a four-dimensional space-time manifold equipped with a parallel transport defined by an affine connection and the associated covariant derivative denoted $\overset{\star}{\nabla}$. Einstein's summation convention is always assumed. Latin indices $a, b, c, ...$ will be used to number the vector fields and forms ("anholonomic coordinates"), indices $i, j, k, ...$ will refer to a coordinate system ("holonomic coordinates").

In a coordinate system x^i, $(i = 0, ..., 3)$ the connection coefficients Δ^k_{ij} are defined by

$$\overset{\star}{\nabla}_j \partial_i = \Delta^k_{ij}\, \partial_k, \tag{II.1}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ are vector fields tangent to the coordinate lines.

[2] Whenever possible, in the references, I am giving the English and the Russian version.

In the literature one meets two opposite conventions of indexing connection coefficients. While in [10, p. 59 (en), p. 66 (ru)], [12, p. 257], [15, p. 261 (en), p. 260 (ru)], [16, p. 256 (en),p. 262 (ru)] we find the same convention as above, the opposite convention, with indices i, j interchanged, is used in [6, p. 278], [7, p. 113], [8, p. 148 (eng), p. 182 (ru)], [4, p. 141 (en), p. 140 (ru)], [9, p. 182 (ru)], [11, p. 271], [13, p. 243], [14, p. 210], [3, p. 9] [17, p. 354 (en), p. 377 (ru)], [18, p. 169].

B. Torsion

Our connection admits *torsion*. Usually the torsion tensor is denoted with the letter T, however The Book is reserving the symbol T for something else (see below), therefore I will denote the torsion tensor by the script letter $\mathscr{T}$. The torsion tensor of *any* affine connection is defined by (cf. [4, p. 133 (en), p. 131 (ru)])

$$\mathscr{T}(X, Y) = \nabla_x Y - \nabla_Y X - [X, Y]. \tag{II.2}$$

Taking $X = \partial_i$, $Y = \partial_j$, with $[\partial_i, \partial_j] = 0$, we obtain local expression for the torsion coefficients $\mathscr{T}^k_{ij}$ in terms of the connection coefficients Δ^k_{ij}

$$\mathscr{T}^k_{ij} = \Delta^k_{ji} - \Delta^k_{ij}. \tag{II.3}$$

Comparing this expression with the formula ((5.20) in The Book, we see that what is called torsion in The Book, and denoted there by the capital Greek letter Ω is just one-half of the usual torsion:

$$\Omega_{ij}^{\cdot\cdot k} = \frac{1}{2}\mathscr{T}^k_{ij},\ \mathscr{T}^k_{ij} = 2\Omega_{ij}^{\cdot\cdot k}. \tag{II.4}$$

C. Curvature

The curvature (I will follow the notation of The Book and denote the curvature tensor by the capital letter S) of *any* connection ∇ is defined by (cf. [4, p. 133 (en), p. 131 (ru)])

$$S(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{II.5}$$

Taking $X = \partial_i, Y = \partial_j, Z = \partial_k$, we obtain the following expression for the curvature tensor coefficients:

$$S^i{}_{jkm} = \Delta^i_{jm,k} - \Delta^i_{jk,m} + \Delta^i_{sk}\Delta^s_{jm} - \Delta^i_{sm}\Delta^s_{jk}, \tag{II.6}$$

where the comma $, k$ indicates partial derivatives with respect to the coordinate x^k. This is Eq. ((5.53)) in The

Book, and it is the same in the standard texts, for instance in [7, p. 117].

D. Contorsion

In a teleparallel theory, like the one that is discussed in The Book, spacetime manifold is endowed not only with an affine connection, but also with a pseudo-Riemannian metric tensor g_{ij} that is preserved by the parallel transport. So we have

$$\overset{\star}{\nabla}_k g_{ij} = 0. \tag{II.7}$$

The metric, on the other hand, induces torsion-free Levi-Civita connection. In The Book the covariant derivative of the Levi-Civita connection is denoted by ∇, and in the following I will respect this notation. The connection coefficients of ∇, usually denoted by $\left\{ {i \atop jk} \right\}$, are denoted by Γ^i_{jk}. I will respect this notation as well. So, from the very definition we have

$$\nabla_j \partial_k = \Gamma^i_{kj}\, \partial_i. \tag{II.8}$$

The Levi-Civita connection has zero torsion, its connection coefficients are symmetric:

$$\Gamma^i_{jk} = \Gamma^i_{kj}. \tag{II.9}$$

The difference of any two affine connections is a tensor. The difference between $\overset{\star}{\nabla}$ and ∇ is called *contorsion*, and in The Book it is denoted by the symbol T :[3]

$$\Delta^i_{jk} = \Gamma^i_{jk} + T^i_{jk}. \tag{(5.28)}$$

III. Error in the formula for the second Bianchi identity

Index-free formulation of the second Bianchi identity satisfied by any affine connection can be found, for instance, in Refs. [4, p. 135, Theorem 5.3 (en), p. 132 (ru)] and [17, p. 360, Eq. (5.22) (en), p. 383 (ru)]. It is expressed as

$$\mathfrak{S}(R(X,Y)Z) = \mathfrak{S}\{\mathscr{T}(\mathscr{T}(X,Y),Z) + (\nabla_X \mathscr{T})(Y,Z)\}, \tag{III.1}$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z. It is easy to express this identity in a coordinate basis in terms of the torsion and curvature tensors. Sulanke [9, p. 189, Eq. (62)] gives such an expansion in an explicit form as follows (I am renaming the indices in to make the comparison with the formula given in The Book easier):

$$\nabla_{[k} \mathscr{T}^i_{jm]} - \mathscr{T}^s_{[kj} \mathscr{T}^i_{m]s} = R^i{}_{kjm}, \tag{III.2}$$

where the square bracket represent the alternation - the same as the symbol $\mathfrak{S}$ in Kobayashi, this time applied to the indices within the bracket. The formula (III.1), when expanded, leads to the same result. Indeed, setting

$$X = \partial_j, Y = \partial_m, Z = \partial_k, \tag{III.3}$$

we get for the LHS

$$\text{LHS} = \mathfrak{S}\{R(\partial_j, \partial_m)\partial_k\} = \mathfrak{S}\{R^i{}_{kjm}\}\partial_i, \tag{III.4}$$

while for the RHS we obtain

$$\begin{aligned} \text{RHS} &= \mathfrak{S}\{\mathscr{T}(\mathscr{T}(\partial_j, \partial_m), \partial_k) + (\nabla_j \mathscr{T})(\partial_m, \partial_k)\} \\ &= \mathfrak{S}\{\mathscr{T}(\mathscr{T}^s_{jm}\partial_s, \partial_k) + \nabla_j \mathscr{T}^i_{mk}\partial_i\} \\ &= \mathfrak{S}\{\mathscr{T}^s_{jm}\mathscr{T}^i_{sk}\partial_i + \nabla_j \mathscr{T}^i_{mk}\partial_i\} \\ &= \mathfrak{S}\{\nabla_j \mathscr{T}^i_{mk} - \mathscr{T}^s_{jm}\mathscr{T}^i_{ks}\}\partial_i. \end{aligned} \tag{III.5}$$

The sign "minus" appear in the last line after we rearranged the indices k, s in the antisymmetric torsion tensor $\mathscr{T}^i_{ks}$ in order for the permuted indices j, m, k to come together.

To be absolutely sure that we have a good formula[4], let us check another classical text, good old Schouten's *Ricci Calculus* [18], quoted in The Book. Owing to the difference of convention used we first make sure that we have the right map from Schouten to The Book. Schouten writes the covariant derivative of a vector field as (cf. [18, p. 124, Eq. (2.3)]:

$$\nabla_\mu v^\kappa = \partial_\mu v^\kappa + \Gamma^\kappa_{\mu\lambda}\, v^\lambda. \tag{III.6}$$

In The Book we find:

$$\overset{\star}{\nabla}_k U^i = U^i_{,k} + \Delta^i_{jk} U^j. \tag{(5.21)}$$

Adapting the indices and comparing we find that

$$\Delta^i_{jk} = \Gamma^i_{kj}, \tag{III.7}$$

[3] Contorsion, also called *contortion*, or *defect* can be expressed in terms of torsion, and torsion in terms of contorsion, cf. e.g. [19]–[21]

[4] In fact, to be *reasonably absolutely* sure one needs to derive the formula oneself, and then check numerically on randomly generated example data, which nowadays, with computers, is not difficult at all. I did it.

where the LHS are the connection coefficients used in The Book, the RHS are the connection coefficients used by Schouten. Next we compare the definitions of torsion. Schouten is using the letter S for his torsion. He defines it as [18, p. 126, Eq. (2.13)]

$$S_{\mu\lambda}^{\cdot\cdot\kappa} = \Gamma_{[\mu\lambda]}^{\kappa}, \tag{III.8}$$

where the symbol of alternation is used the same way as in The Book, thus

$$\Gamma_{[\mu\lambda]}^{\kappa} = \frac{1}{2}(\Gamma_{\mu\lambda}^{\kappa} - \Gamma_{\lambda\mu}^{\kappa}). \tag{III.9}$$

The Book defines torsion Ω as

$$\Delta_{[ij]}^{k} = -\Omega_{ij}^{\cdot\cdot k}. \tag{(5.20)}$$

Since $\Omega_{ij}^{\cdot\cdot k} = -\Omega_{ji}^{\cdot\cdot k}$, we have

$$\Omega_{ji}^{\cdot\cdot k} = \Delta_{[ij]}^{k} = \Gamma_{[ji]}^{k} = S_{ji}^{\cdot\cdot k}, \tag{III.10}$$

Therefore *torsion Ω in The Book is identical to torsion S in Schouten*. Schouten proves the second Bianchi identity in the following form [18, p. 144, Eq. (5.2)]

$$R_{[\nu\mu\lambda]}^{\cdot\cdot\cdot\kappa} = 2\nabla_{[\nu}S_{\mu\lambda]}^{\cdot\cdot\kappa} - 4S_{[\nu\mu}^{\cdot\cdot\rho}S_{\lambda]\rho}^{\cdot\cdot\kappa}. \tag{III.11}$$

Setting $\kappa \to i, \nu \to k, \mu \to j, \lambda \to m, \rho \to s, R = 0, S \to \Omega, \nabla \to \overset{\star}{\nabla}$, we again arrive at

$$\overset{\star}{\nabla}_{[k}\,\Omega_{jm]}^{\cdot\cdot i} - 2\,\Omega_{[kj}^{\cdot\cdot s}\,\Omega_{m]s}^{\cdot\cdot i} = 0. \tag{III.12}$$

In order to compare the Bianchi identity with the formula from The Book, we should always set $R = 0$, as this is the main assumption in The Book that describes the teleparallel case, with identically vanishing curvature. The formula (5.60) in Proposition 5.6 in The Book reads:

$$\overset{\star}{\nabla}_{[k}\Omega_{jm]}^{\cdot\cdot i} + 2\Omega_{[kj}^{\cdot\cdot s}\Omega_{m]s}^{\cdot\cdot i} = 0. \tag{(5.60)}$$

Using Eq. (II.4) we see that the sign in the formula is wrong. It is impossible to trace the exact origin of this wrong sign since the proof of the formula in Proposition 5.6 is incomplete.

A. Why bother?

The Reader may ask why do I pay so much attention to an error in a formula? And indeed, I would probably pay much less attention if not for the fact that the same error is being repeated in other publications. Randomly checking I have found the same wrong formula in [22, Eq. (1.60)], [23, Eq. (60)], [24, Eq. (31)], [25, Eq. (33)]. Of course one may ask: perhaps the sign does not matter? But if the sign in a mathematical formula does not matter, then the formula itself does not matter. And if so, then why write it at all?

IV. Errors in the treatment of differential forms and anholonomic frame

Sections 5.4 and 5.5 of The Book are difficult to analyze as they contain internal *mathematical inconsistencies*. The titles of these sections are: *Formalism of external forms and the matrix treatment of Cartan's structural equations of the absolute parallelism geometry*, and A_4 *geometry as a group manifold. Killing-Cartan metric.* It seems that the internal inconsistencies in these two sections follow from the fact that the connection coefficients in an anholonomic frame are defined not in the way it is done in differential geometry. Here I will explain what the problems with these two sections are, I will point out the contradictions that need to be resolved in order for the content of these sections to make any mathematical sense at all. I will also explain the problems in some detail with the hope that it may help the author to fix these problems in the future.

The method of analyzing a general affine connection in an anholonomic frame is described, for instance, in Schouten's *Ricci Calculus* [18], Ch. III,§9, *Linear connexions expressed in anholonomic coordinates*, Ch. III,§10, *Cartan's symbolical method used for connexions*. I will first describe the method, adapting the notation and terminology of Ref. [18] to the one used in The Book.

An anholonomic frame in a spacetime manifold consists of four linearly independent vector fields e_a, $(a = 0, ..., 3)$ that can be expressed as linear combinations of the vector fields ∂_i tangent to coordinates lines of a (holonomic) coordinate system x^i:

$$e_a = e_a^i\,\partial_i. \tag{IV.1}$$

Then there is also the dual coframe (tetrad) e^a of differential forms[5]

$$e^a = e^a_i \, dx^i. \tag{IV.2}$$

The duality is expressed through the relations:

$$e^i_a e^b_i = \delta^b_a, \quad e^i_a e^a_j = \delta^i_j. \tag{IV.3}$$

Any vector field v can be expressed in terms of either the holonomic basis ∂_i or anholonomic basis e_a:

$$v = v^i \, \partial_i = v^a \, e_a, \tag{IV.4}$$

with

$$v^a = e^a_i v^i, \quad v^i = e^i_a \, v^a. \tag{IV.5}$$

Similarly for any differential form w

$$w = w_i \, dx^i = w_a \, e^a, \tag{IV.6}$$

$$w_a = e^i_a \, w_i, \quad w_i = e^a_i \, w_a. \tag{IV.7}$$

Given *any* affine connection ∇ the connection coefficients Γ^a_{bc} with respect to the anholonomic frame are defined by the formula

$$\nabla_c \, e_b = \Gamma^a_{bc} \, e_a, \tag{IV.8}$$

where

$$\nabla_c = \nabla_{e^i_c \partial_i} = e^i_c \nabla_i. \tag{IV.9}$$

One can also use the connection coefficients Γ^a_{bi} defined by

$$\nabla_i \, e_b = \Gamma^a_{bi} \, e_a, \tag{IV.10}$$

with

$$\Gamma^a_{bc} = e^i_c \, \Gamma^a_{bi}, \quad \Gamma^a_{bi} = e^c_i \, \Gamma^a_{bc}. \tag{IV.11}$$

Thus, given a fixed anholonomic frame, we have connection one-forms

$$\Gamma^a_b = \Gamma^a_{bc} \, e^c = \Gamma^a_{bi} \, dx^i. \tag{IV.12}$$

From the definition (IV.8) we derive the formula for the covariant derivative of a vector field v expressed in anholonomic frame:

$$\begin{aligned}(\nabla_a v)^b &= (\nabla_a v^c \, e_c)^b = \partial_a v^c \, (e_c)^b + v^c \, \Gamma^b_{ca} \\ &= \partial_a \, v^b + \Gamma^b_{ca} \, v^c,\end{aligned} \tag{IV.13}$$

with (IV.8) and (IV.13) being equivalent. This is the formula (9.1) in [18, p. 169].

[5]In The Book, for unknown reasons, the terminology is reversed: e_a are called *covectors*, e^a are called *vectors*.

From the definitions of the connection coefficients we can find the relation between holonomic and anholonomic coefficients as follows

$$\nabla_i \, e_a = \Gamma^b_{ai} \, e_b, \tag{IV.14}$$

$$\nabla_i \, e_a = \nabla_i \, (e^j_a \, \partial_j) \tag{IV.15}$$

$$= e^j_{a,i} \, \partial_j + e^k_a \, \Gamma^j_{ki} \, \partial_j \tag{IV.16}$$

$$= (e^j_{a,i} + e^k_a \, \Gamma^j_{ki}) \, e^b_j \, e_b. \tag{IV.17}$$

Therefore

$$\Gamma^b_{ai} = e^b_j \, e^j_{a,i} + e^k_a \, e^b_j \, \Gamma^j_{ki} = -e^j_a \, e^b_{j,i} + e^k_a \, e^b_j \, \Gamma^j_{ki}. \tag{IV.18}$$

This is, in essence, Eq. (9.2) in Schouten [18, p. 169]. Writing in terms of differential forms, and renaming the indices the above formula reads

$$\Gamma^a_b = e^a_i \, d\, e^i_b + e^k_b \, e^a_l \Gamma^l_k. \tag{IV.19}$$

The above explains how connection coefficients in anholonomic frames are dealt with in differential geometry.[6] I have quoted here only Ref. [18], but the same content can be found in any other text discussing this subject, e.g. [26, p. 466, Eq. (3.3), second line], [27, p. 102, Eq. (7.1) (en), p. 150 (ru)], [10, p. 59, Eq. (38.4) (en), p. 66 (ru)].

After these introductory comments let us return to the content of sections 5.4 and 5.5 of The Book. For some reason the connection coefficients $\Delta^a{}_b$ there are defined not as in the standard text on differential geometry, as given in Eqs. (IV.18), (IV.19). They are defined using *only the first part of the full formula* (IV.18), (IV.19):

$$\Delta^a{}_b = e^a_i \, d\, e^i_b, \tag{(5.65)}$$

that is: $\Delta^a_{bj} = e^a_i \, e^i_{b,j}$. And *this is probably one of the main reasons for which the content of these two sections is confusing and contradictory*, as I will explain now.[7]

[6]All of that can be translated into the language of principal connections on the bundle of linear frames (or its reduction to the bundle of orthogonal frames), which is the modern way of discussing Cartan's formalism; but such a formulation would not add anything of real importance here. I am trying to keep the discussion at the almost elementary level, often used by physicists and sufficient for our purpose.

[7]Of course we are allowed to define the quantities that, for some reason, we are interested in, any way we want. But then we should not be surprised that the properties and formulas that are used in the literature for the quantities defined differently do not automatically apply.

A. Error in the definition of absolute parallelism geometry

The first odd statement we meet in Ch. 5.4 of The Book reads as follows

> *By definition, a space has a geometry of absolute parallelism, if the 2-form of Cartanian torsion S^a and the 2-form of the Riemann-Christoffel curvature $S^b{}_a$ of this space vanish*

$$S^a = 0, \qquad ((5.71))$$

$$S^b{}_a = 0. \qquad ((5.72))$$

One of the main references in The Book is (somewhat outdated textbook) [26]. There, at the bottom of p. 485 (ru) we find the statement that is standard in differential geometry (English translation from Russian edition):

> *If $r = n$, [i.e. if the number of parallel vector fields is equal to the dimension of the manifold] then $R^k_{hlm} = 0$ and the space has a vanishing curvature. One says, that such a space is a space of absolute parallelism.*

To state that also the torsion should vanish, as it is stated in The Book, is, at least, odd. The confusion probably stems from the "original" definition of the connection coefficients, as I have explained it above. Simply copying the standard definition of the torsion in terms of connection coefficients, and using it with a different definition of these coefficients, surely must lead to confusion.

Remark 4.1: In spite of the strange definition of the connection coefficients it will be useful to have some geometrical interpretation of $\Delta^a{}_b$. In fact this is simple by looking again at the formula (IV.18) and noticing what has been omitted in the definition of $\Delta^a{}_b$, namely Γ^j_{ki} has been set to zero. It follows that $\Delta^a{}_b$ can be interpreted as the true connection coefficients (referenced to the moving frame e^a) of the unique connection that has zero connection coefficients in the coordinate frame ∂_i. In [12, p. 318, Example 1] such a connection is called *standard* (with respect to the coordinate system x^i), and is denoted by δ.

B. Error in contorsion transformation rules

Let us now move to another source of the confusion, the definition of the "contorsion coefficients" $T^a{}_b$, and Levi-Civita connections coefficients $\Gamma^a{}_b$, the main objects appearing in the two sections under analysis. Quoting from The Book:

> *Considering (5.28), we will represent 1-form $\Delta^a{}_b$ as the sum*

$$\Delta^a{}_b = \Gamma^a{}_b + T^a{}_b. \qquad ((5.77))$$

While the LHS of this last equation has been defined (though in a nonstandard way), on the RHS we have two undefined objects: $\Gamma^a{}_b$ and $T^a{}_b$. Since they are undefined, we can only try to guess what can be their meaning, and analyze the following formulas in order to check whether our guess is right or not. Fortunately we can find a formula for T four pages later, where it is defined as

$$T^a{}_{bk} = e^a_i \, e^j_b \, T^i_{jk} = e^j_b \, \nabla_k \, e^a_j, \qquad ((5.113))$$

where ∇ stands for the Levi-Civita connection, and the equality holds under the assumption that $\overset{\star}{\nabla} e_a = 0$. The first equality is a natural one. Contorsion, as a difference of two connections, is a tensorial object, and the first equality is the correct way to express it in an anholonomic frame. The second equality holds only for a parallel frame. And yet it is the second formula, namely

$$T^a{}_{bk} = e^j_b \, \nabla_k \, e^a_j \qquad ((5.113))$$

that seems to be exploited in Eq. ((5.88)) that is supposed to provide the transformation character of T under gauge transformations:

$$e^{a'}_m = \Lambda^{a'}_a \, e^a_m. \qquad ((5.87))$$

Eq. ((5.88)) states:

$$T^{a'}_{b'k} = \Lambda^{a'}_a \, T^a_{bk} \, \Lambda^b_{b'} + \Lambda^{a'}_a \, \Lambda^a_{b',k}. \qquad ((5.88))$$

The point is that even with rather strange definition ((5.113)), the formula ((5.88)) is wrong - it has a wrong sign. The verification is straightforward:

$$\begin{aligned} T^{a'}_{b'k} &= e^j_{b'} \nabla_k e^{a'}_j = \Lambda^b_{b'} e^j_b \nabla_k (\Lambda^{a'}_a \, e^a_j) \\ &= e^j_b e^a_j \, \Lambda^b_{b'} \Lambda^{a'}_{a,k} + e^j_b \Lambda^b_{b'} \Lambda^{a'}_a \nabla_k \, e^a_j \\ &= \Lambda^a_{b'} \Lambda^{a'}_{a,k} + \Lambda^{a'}_a \Lambda^b_{b'} T^a_{bk} \\ &= \Lambda^{a'}_a \, T^a_{bk} \, \Lambda^b_{b'} - \Lambda^{a'}_a \, \Lambda^a_{b',k}, \end{aligned} \qquad (IV.20)$$

where we have used $\Lambda^a_{b'}\Lambda^{a'}_{a,k} = -\Lambda^{a'}_a\Lambda^a_{b',k}$, owing to the fact that $\Lambda^a_{b'}\Lambda^{a'}_a = \delta^{a'}_{b'}$. Thus, in Eq. ((5.88)) the sign should be "minus" instead of "plus". I have traced the same error, sometimes with different names for the indices, in Refs. [22, p. 27, Eq. (1.88)], [23, Eq. (88)], [28, Eq. 18], [24, Eq. (33)].

V. ERRORS IN USING CARTAN'S METHOD

The wrong sign in Eq. ((5.88)) is not a big deal. Since the formula will probably never be used, it will have little or no consequence at all. But now I coming to a much more serious problem. It all starts with the definition ((5.65)) of $\Delta^a{}_b$. As I have already noted, this is not how connection coefficients with respect to a moving frame are defined in differential geometry. While everybody can define quantities that she/he is interested in any way she/he wishes, there is a price attached to this freedom. As I will show now, in this case, the price happens to be rather high.

Probably the best way to demonstrate errors in mathematical statements is by providing simple but convincing counterexamples. I will provide two such counterexamples, and I will show that the whole idea of treatment of differential forms and anholonomic frame, expanded in chapters 5.4 and 5.5, of The Book is wrong from the very beginning.[8] In particular, *with the definitions given as in The Book*, the formulas for "Cartan's structural equations and Bianchi identities for A_4 geometry" listed in a box on p. 23 of The Book as

$$\boxed{\begin{array}{rcll} de - e\wedge T &=& 0, & ((A)) \\ R + dT - T\wedge T &=& 0, & ((B)) \\ R\wedge e\wedge e\wedge e &=& 0, & ((C)) \\ dR + R\wedge T - T\wedge R &=& 0. & ((D)) \end{array}} \tag{V.1}$$

- they all *do not hold.*

A. Counterexample 1: The "navigator connection"

As the first counterexample let us take one of the simplest and oldest realization of teleparallelism given as an exercise in Schouten's classical text [18, p. 143, Exercise III.4,1].[9] The exercise, a mathematical toy model, is formulated there as follows - cf. Fig. 1:

> "A man is moving on the surface of the earth always facing one definite point, say Jerusalem or Mekka or the North pole. Prove that this displacement is semi-symmetric and metric and compute S_λ."

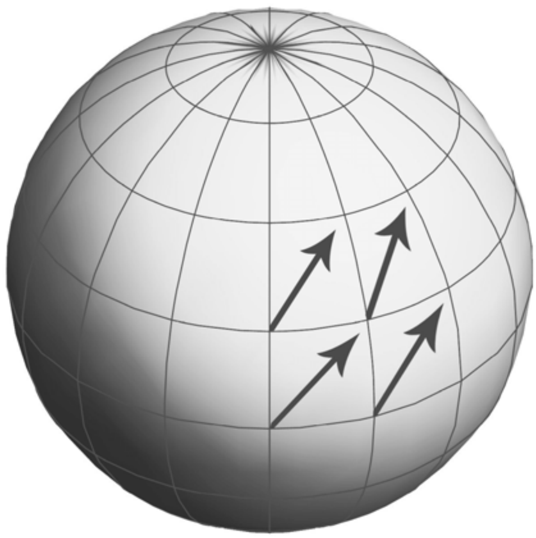

Figure 1. The navigator connection: parallelism defined as on the Mercator map.

Nakahara [14, p. 216-219] discusses this example in some detail in section 7.3.2 entitled "*Geometrical meaning of the Riemann tensor and the torsion tensor*", Example 7.11, calculating the torsion. Fernández and W. A. Rodrigues calculate even more in Appendix B, of Ref. [29]: "*Levi-Civita and Nunes Connctions on* $\overset{\circ}{S}{}^2$". The autoparallel geometry connection, called in [29] *navigator* or *Nunes* connection, is defined as follows [14]: Suppose we are navigating on the surface of the Earth. We *define* a vector to be parallel transported if the angle between the vector and the latitude is kept fixed during the navigation. Let us first define the holonomic

[8] Mathematica notebooks containing the calculations for these two examples can be downloaded from http://arkadiusz-jadczyk.org/Navigator.nb, http://arkadiusz-jadczyk.org/S3xR.nb

[9] This toy model is unphysical, it is two-dimensional. As we will see in the second counterexample adding another two dimensions will not change anything important in the discussion.

coordinates, frame, coframe and the metric.

Coordinates $(x^1, x^2) = (\theta, \phi)$, $0 < \theta < \pi)$, $0 < \phi < 2\pi)$. Holonomic basis: ∂_i. Moving frame

$$e_1 = \partial_1,\ e_2 = \frac{1}{\sin x^1}\,\partial_2, \tag{V.2}$$

$$e_1{}^1 = 1,\ e_1{}^2 = 0,\ e_2{}^1 = 0,\ e_2{}^2 = \frac{1}{\sin x^1}. \tag{V.3}$$

Coframe

$$e^1{}_1 = 1,\ e^1{}_2 = 0,\ e^2{}_1 = 0,\ e^2{}_2 = \sin x^1. \tag{V.4}$$

1) Geometric quantities in the coordinate basis: Metric, which coincides with the standard metric on the surface of the unit sphere:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 x^1 \end{bmatrix}, \quad [g^{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{sin^2 x^1} \end{bmatrix}. \tag{V.5}$$

Nonzero (holonomic) Levi-Civita connection coefficients Γ^i_{jk}:

$$\Gamma^1_{22} = -\cos x^1 \sin x^1,\ \Gamma^2_{12} = \Gamma^2_{21} = \cot x^1. \tag{V.6}$$

Nonzero (holonomic) Riemann curvature coefficients $R^i{}_{jkm}$:

$$\begin{aligned} R^1{}_{212} &= -R^1{}_{221} = \sin^2 x^1, \\ R^2{}_{121} &= -R^2{}_{112} = 1. \end{aligned} \tag{V.7}$$

Nonzero (holonomic) coefficients of the absolute parallelism connection $\Delta^i{}_{jk}$:

$$\Delta^2{}_{21} = \cot x^1. \tag{V.8}$$

Nonzero (holonomic) torsion $\mathscr{T}^i{}_{jk} = \Delta^i_{kj} - \Delta^i_{jk}$:

$$\mathscr{T}^2{}_{12} = -\mathscr{T}^2{}_{21} = \cot x^1. \tag{V.9}$$

Nonzero (holonomic) contorsion $T^i{}_{jk}$:

$$T^1{}_{22} = \cos x^1 \sin x^1, \quad T^2{}_{12} = -\cot x^1. \tag{V.10}$$

2) $\Delta^a{}_b$, $\Gamma^a{}_b$ and $R^a{}_b$ in the parallel (anholonomic) frame, calculated as in The Book (wrong way): The frame is anholonomic. We have

$$[e_i, e_j] = c^k_{ij}\, e_k, \quad c^2_{21} = -c^2_{12} = \cot x^1. \tag{V.11}$$

We calculate $[\Delta^a{}_b]_i$ according to Eq. ((5.67)) of The Book. The result is

$$[\Delta]_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\cot x^1 \end{bmatrix}, \quad [\Delta]_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{V.12}$$

Then we calculate

$$[T^a{}_b]_i$$

from ((5.113)), and $\Gamma^a{}_b = \Delta^a{}_b - T^a{}_b$ from ((5.77)):

$$[T]_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad [T]_2 = \begin{bmatrix} 0 & \cos x^1 \\ -\cos x^1 & 0 \end{bmatrix}. \tag{V.13}$$

$$[\Gamma]_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\cot x^1 \end{bmatrix}, [\Gamma]_2 = \begin{bmatrix} 0 & \cos x^1 \\ -\cos x^1 & 0 \end{bmatrix}. \tag{V.14}$$

Finally we can calculate "the Riemann tensor" $R^a{}_{bij}$ from the formula ((5.78)). The nonzero components are

$$[R^a{}_b]_{12} = -[R^a{}_b]_{21} = \begin{bmatrix} 0 & -\cos 2x_1 \csc x_1 \\ -\csc x_1 & 0 \end{bmatrix}. \tag{V.15}$$

What is wrong? Our matrices should be in the Lie algebra $so(2)$ of the rotation group - they should be antisymmetric. But $[\Gamma]_1$ and $[R^a{}_b]_{12}$ are not antisymmetric!

B. Counterexample 2: Absolute parallelism on the sphere S^3 and Einstein's static universe

This counterexample is related to Einstein's static universe geometry. We formulate it as follows: The only parallelizable spheres are S^1, S^3 and S^7. We are interested in S^3, the group manifold of the Lie group $SU(2)$, as it the space part of the four-dimensional spacetime manifold $S^3 \times \mathbb{R}$. Nakahara [14, p. 220] describes the parallelism of S^3 using left-invariant vector fields of the natural action of unit quaternions. The group $SU(2)$ is parametrized as

$$\begin{gathered} U(Z_1, Z_2) = \begin{bmatrix} Z_1 & -\bar{Z}_2 \\ Z_2 & \bar{Z}_1 \end{bmatrix}, \\ Z_1 = X_1 + iX_2,\ Z_2 = X_3 + iX_4, \\ X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1. \end{gathered} \tag{V.16}$$

Let us introduce hyperspherical coordinates x_1, x_2, x_3, with $x_3 \in (0, \pi/2)$, $x_1, x_2 \in [0, 2\pi)$, as follows

$$\begin{aligned} X_1 &= \cos x_1 \cos x_3, \\ X_2 &= \sin x_1 \cos x_3, \\ X_3 &= \cos(x_1 + x_2) \sin x_3, \\ X_4 &= \sin(x_1 + x_2) \sin x_3, \end{aligned} \tag{V.17}$$

$$\begin{aligned} x_1 &= \arctan X_2/X_1, \\ x_2 &= \arctan \frac{X_1 X_4 - X_2 X_3}{X_1 X_3 + X_2 X_4}, \\ x_3 &= \arctan \sqrt{\frac{X_3^2 + X_4^2}{X_1^2 + X_2^2}}. \end{aligned} \tag{V.18}$$

Then we define four vector fields $e_a,$, three of them are the left-invariant (fundamental) vector fields on $S^3 \approx SU(2)$:

$$\begin{aligned} e_0 &= \partial_0, && \text{(V.19)}\\ e_1 &= -\cos(2x_1+x_2)\tan x_3\,\partial_1 \\ &\quad +\cos(2x_1+x_2)\csc x_3\sec x_3\,\partial_2 \\ &\quad +\sin(2x_1+x_2)\,\partial_3, && \text{(V.20)}\\ e_2 &= -\sin(2x_1+x_2)\tan x_3\,\partial_1 \\ &\quad +\sin(2x_1+x_2)\csc x_3\sec x_3\,\partial_2 \\ &\quad -\cos(2x_1+x_2)\,\partial_3, && \text{(V.21)}\\ e_3 &= \partial_3. && \text{(V.22)} \end{aligned}$$

Trajectories of vector fields e_1, e_2 are shown in Figs. 2,3.

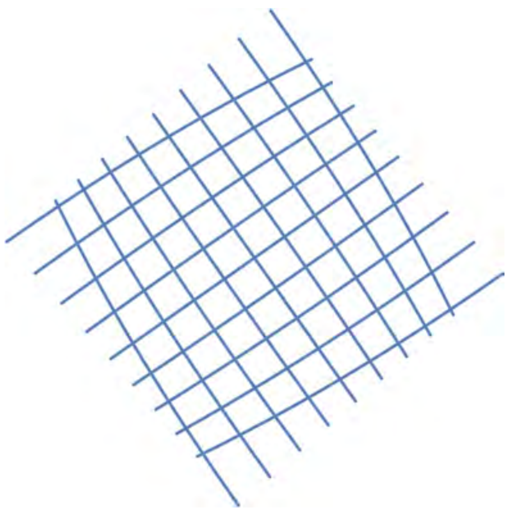

Figure 2. Trajectories of e_1 and e_2 near the point $(0,0,0,-1)$ on S^3 projected stereographically into $\mathbb{R}^3$, as viewed from a point on the x axis in $\mathbb{R}^3$. They look like a regular coordinate grid.

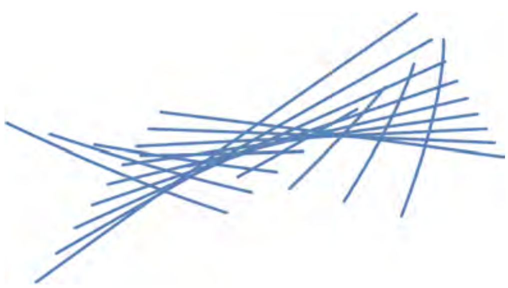

Figure 3. The same trajectories viewed from another angle. It can be seen that the rectangles formed by the field lines do not close - because of the torsion.

Remark 5.1: The coordinates (x_1, x_2, x_3) are not very convenient for describing the geometry of left-invariant vector fields - the formulas below are not simple at all. In fact these coordinates are well adapted to the geometry of right-invariant vector fields. Nevertheless the advantage of our coordinates is in the fact that the coordinate lines of x_1 are geodesics - cf. Fig 4.

Then

$$[e_1, e_2] = 2\,e_3,\ [e_2, e_3] = 2\,e_1,\ [e_3, e_1] = 2\,e_2, \qquad \text{(V.23)}$$

all other Lie brackets being zero.[10]

Metric; signature $(1,-1,-1,-1)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0, \\ 0 & -1 & -\sin^2 x_3 & 0 \\ 0 & -\sin^2 x_3 & -\sin^2 x_3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \qquad \text{(V.24)}$$

Nonzero (holonomic) Levi-Civita connection coefficients Γ^i_{jk}:

$$\begin{aligned} &\Gamma^1_{13} = \Gamma^1_{31} = -\tan x_3, \quad \Gamma^2_{23} = \Gamma^2_{32} = \cot x_3, \\ &\Gamma^2_{13} = \Gamma^2_{31} = \sec x_3 \csc x_3, \\ &\Gamma^3_{12} = \Gamma^3_{21} = \Gamma^3_{22} = -\sin x_3 \cos x_3. && \text{(V.25)} \end{aligned}$$

Nonzero (holonomic) components of the Riemann curvature tensor of the Levi-Civita connection: ${R^i}_{jkm} = \nabla_k \Gamma^i_{jm} - \nabla_m \Gamma^i_{jk}$:

$$\begin{aligned} {R^1}_{112} &= -{R^1}_{121} = {R^1}_{212} = -{R^1}_{221} = -{R^2}_{212} \\ &= {R^2}_{221} = -{R^3}_{123} = {R^3}_{132} = -{R^3}_{213} \\ &= -{R^3}_{223} = {R^3}_{231} = {R^3}_{232} = \sin^2 x^3, \\ {R^1}_{313} &= -{R^1}_{331} = -{R^2}_{112} = {R^2}_{121} = {R^2}_{323} \\ &= -{R^2}_{332} = -{R^3}_{113} = {R^3}_{131} = 1. && \text{(V.26)} \end{aligned}$$

Nonzero (holonomic) coefficients of the connection of the absolute parallelism Δ^i_{jk}:

$$\begin{aligned} &{\Delta^1}_{23} = -{\Delta^1}_{32} = -\frac{1}{2}{\Delta^1}_{31} = \tan x_3, \\ &{\Delta^2}_{23} = \cot x_3 - \tan x_3, \\ &{\Delta^2}_{32} = \frac{1}{2}{\Delta^2}_{31} = \sec x_3 \csc x_3, \\ &{\Delta^3}_{21} = -\sin 2x_3, \\ &{\Delta^3}_{22} = -\sin x_3 \cos x_3. && \text{(V.27)} \end{aligned}$$

[10]The discussion in [14, p. 220] is somewhat misleading. What he calculates are parts of the Lie brackets, and not the connection coefficients as claimed. As the result his torsion has a wrong sign.

Nonzero (holonomic) coefficients $\mathscr{T}^i{}_{jk}$ of the torsion of the connection of the absolute parallelism Δ^i_{jk}: Nonzero (holonomic) torsion: $\mathscr{T}^i{}_{jk} = \Delta^i{}_{kj} - \Delta^i{}_{jk}$:

$$\begin{aligned}
&\mathscr{T}^1{}_{31} = -\mathscr{T}^1{}_{23} = -\mathscr{T}^1{}_{13} = \mathscr{T}^1{}_{32} = 2\tan x^3,\\
&\mathscr{T}^2{}_{13} = -\mathscr{T}^2{}_{31} = 2\sec x^3 \csc x_3,\\
&\mathscr{T}^2{}_{32} = -\mathscr{T}^2{}_{23} = \cot x_3 - \sec x_3 \csc x_3 - \tan x_3,\\
&\mathscr{T}^3{}_{21} = -\mathscr{T}^3{}_{12} = \sin 2x_3
\end{aligned} \tag{V.28}$$

Nonzero (holonomic) coefficients $T^i{}_{jk}$ of the contorsion: $T^i{}_{jk} = \Delta^i{}_{jk} - \Gamma^i_{jk}$:

$$\begin{aligned}
&T^1{}_{13} = T^1{}_{23} = -T^1{}_{31} = -T^1{}_{32} = -T^2{}_{23} = T^2{}_{32}\\
&\qquad = \tan x^3,\\
&T^2{}_{13} = -T^2{}_{31} = -\sec x_3 \csc x_3,\\
&T^3{}_{12} = -T^3{}_{21} = \sin x_3 \cos x_3
\end{aligned} \tag{V.29}$$

Remark 5.2: Notice that the symmetric part of the contorsion tensor vanishes: $T^i_{(jk)} = 0$. Therefore the geodesics of the Levi-Civita connection coincide with autoparallels of the connection $\overset{\star}{\nabla}$ of absolute parallelism generated by the left-invariant vector fields - cf. Fig 4. In fact these geodesics are trajectories of one-parameter subgroups of the right action of $SU(2)$ on S^3. Left actions, on the other hand, generate Killing vector fields of the geometry.

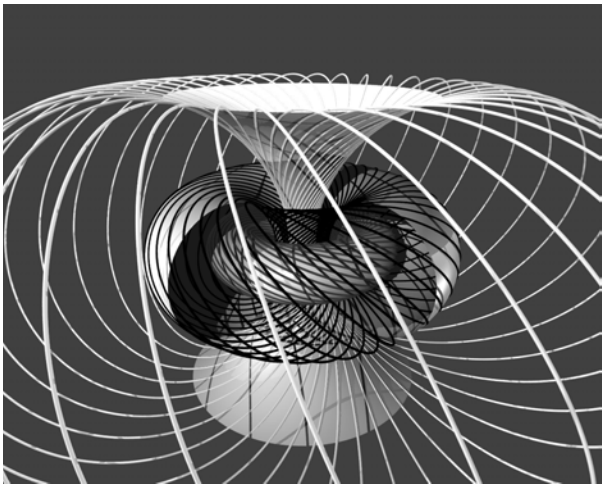

Figure 4. Stereographic projection of surfaces of constant coordinate x_3 for $x_3 = \pi/8, \pi/4, 3\pi/8$. The circles are geodesics - coordinate lines of x_1. Each circle represents one fiber of the "Hopf fibration" $S^3 \to S^2$. For more details, as well as for the relation to quantum spin $1/2$ states see Ref. [30, p. 60-68]

1) $\Gamma^a{}_b$ and $R^a{}_b$ in the parallel (anholonomic) frame, calculated as in The Book: I will not give the result of calculations of the four matrices $[\Gamma^a{}_b]_i$, $(i = 0, ..., 3$ and sixteen matrices $[R^a{}_b]_{ij}$ calculated according to the rules given in The Book. It would take two pages and the results are meaningless anyway. Correct, simple values are in Eqs. (V.31), (V.32), V.33). Let me just give, as examples, one component of each, *that should be zero for any antisymmetric matrix, but which is not*:

$$\begin{aligned}
&[\Gamma^1{}_1]_3 = -\cos^2(2x_1 + x_2)\cos(2x_3)\csc(2x_3)\sec(2x_3),\\
&[R^1{}_1]_{13} = -2\cot(2x_3)\sin(2x_1 + x_2)
\end{aligned} \tag{V.30}$$

C. Why is it bad?

Why is it bad, and is it all bad? Well, the formula ((5.513)) is good, but that is about all that is good in these sections of The Book, where we are supposed to replace the holonomic, non-orthonormal basis by a anholonomic but orthonormal basis and, in this way, transform our quantities of interest so that they have values in the Lie algebra of the Lorentz group $O(3,1)$. This is the whole trick behind the formulations of gravity theories as gauge theories of the Lorentz or of the Poincaré group - the idea that The Book aims to address in these sections. Yet *neither the coefficients $\Delta^a{}_b$ of the teleperallel connection, nor the coefficients $\Gamma^a{}_b$ of the Levi-Civita connection (and, in consequence, the coefficients $R^a{}_b$ of the Riemann curvature of the Levi-Civita connection) take values in the Lie algebra of the Lorentz group*. In the two counterexamples, with the navigator connection and with the parallelizable S^3, it is seen from the form of the matrices $[\Gamma^a{}_b]$ and $R^a{}_b$. These matrices, as infinitesimal generators of rotations, should be *antisymmetric*, but they are not. As the result *none of the equations $((B))$, $((C))$, $((D))$ in the table V.1 is true*. $((A))$ is also wrong, though, as I explain below, for a different reason.

D. How should it be done?

For the "navigator connection: $\Gamma^a{}_{bi}$ should be calculated from the formula (IV.18), with the result

$$[\Gamma^a{}_b]_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \quad [\Gamma^a{}_b]_2 = \begin{bmatrix} 0 & -\cos x^1 \\ \cos x^1 & 0 \end{bmatrix}. \tag{V.31}$$

Then for the corresponding curvature $R^a{}_{bij}$ we would get

$$[R^a{}_b]_{12} = -[R^a{}_b]_{21} = \begin{bmatrix} 0 & \sin x^1 \\ -\sin x^1 & 0 \end{bmatrix}. \qquad \text{(V.32)}$$

For the $S^3 \times \mathbb{R}$ geometry: Nonzero anholonomic Levi-Civita connection $\Gamma^a{}_{bc}$ and its Riemann curvature tensor (only space part given, time part is trivial) with values in the Lie algebra $so(3)$ are

$$\begin{aligned}
[\Gamma^a{}_b]_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \ [\Gamma^a{}_b]_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\
[\Gamma^a{}_b]_3 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ [R^a{}_b]_{12} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
[R^a{}_b]_{13} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \ [R^a{}_b]_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.
\end{aligned} \qquad \text{(V.33)}$$

Notice that, additionally, $R^a{}_{bcd} = -R^a{}_{bdc}$.

Both would be antisymmetric, as it should be. $\Delta^a{}_b$ *should be set identically zero (we are in a parallel frame!)*. After that structure equations and Bianchi identities can be simply copied from the numerous textbooks.

E. What else is bad?

Now, knowing what is $\Delta^a{}_b$ and what is $T^a{}_b$ we can deduce the definition of $\Gamma^a{}_b = \Delta^a{}_b - T^a{}_b$. Elementary calculation leads to

$$\Gamma^a{}_{bk} = e^a_i (e^j_b \, \Gamma^i_{jk} + 2\, e^i_{b,k}). \qquad \text{(V.34)}$$

If the sign in the definition of $\Delta^a{}_b$ would be opposite, the ugly second term would disappear, but even then, comparing the above with the correct formula (IV.18), we would see that the definition of $\Gamma^a{}_b$ has been crippled. This time, however, the first term of the correct definition would be missing.

It is rather surprising that the sign in the simple statement (A) is also wrong. The statement in The Book (p.23, bottom) reads:

$$de - e \wedge T = 0, \qquad \text{(V.35)}$$

which is then expanded in Proposition 5.8 to

$$de^a - e^c \wedge T^a{}_c = 0. \qquad \text{((5.84))}$$

We have

$$(de^a)_{ij} = \partial_i \, e^a_j - \partial_j \, e^a_i, \qquad \text{(V.36)}$$

and, using the definition ((5.113))

$$\begin{aligned}
(e^c \wedge T^a{}_c)_{ij} &= e^c_i \, T^a{}_{cj} - e^c_j \, T^a{}_{ci} \\
&= e^c_i \, e^k_c \, \nabla_j \, e^a_k - e^c_j \, e^k_c \, \nabla_i \, e^a_k \\
&= \nabla_j \, e^a_i - \nabla_i \, e^a_j = \partial_j \, e^a_i - \partial_i \, e^a_j.
\end{aligned} \qquad \text{(V.37)}$$

Therefore the correct formula (A) is

$$de^a + e \wedge T = 0. \qquad \text{(V.38)}$$

Again we have error in the sign.

VI. CONCLUSIONS

Spacetime torsion and its interaction with spinning matter is an old subject, but today as much relevant as it was eighty years ago, perhaps even more (for a modern introduction cf. [31]); it may become even more important in the future. In a recent publication [32], entitled "Prospects of detecting spacetime torsion" Puetzfeld and Obukhov notice that " One surprising feature of nonminimal theories turns out to be their potential sensitivity to torsion of spacetime even in experiments with ordinary (not microstructured) test matter."

However, the mathematics of torsion is more advanced and more subtle than what is needed in the standard general relativity. The mathematician Elie Cartan, in his letter of 1932 to Albert Einstein wrote [33, p. 231]:

> " *Cher et illustre Maître*,
> Your letter has filled me with both joy and confusion. Of course I take pleasure in our little exchange and, if it were up to me, I would willingly become young again, if not to give you lessons, at least to follow, better than I can now, all the marvelous things being done in physics. (...)"

In conclusion: In these notes I analyzed the mathematical part of G. I. Shipov's "A Theory of Physical Vacuum", Ch. 5, and pointed out multiple errors there. I have also indicated the proper way of dealing with this subject. I hope that these comments can be seen as complementary to a rather superficial review by V. A. Rubakov [34], who wrote that The Book contains "well known geometrical constructions" and "plentitude of formulas", but who, apparently, did not even try to understand what these formulas are about. As a result evident mathematical inconsistencies escaped his attention. Modern theoretical physics requires advanced mathematics, and anyone using such mathematical tools should, first of all, have a clear understanding of the meaning of mathematical operations and formulas. Otherwise confusion and misinterpretation will prevail.

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